A Bi-Objective Decomposition Method for Solving the Bi-Objective Multi-Commodity Minimum Cost Flow Problem

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Abstract

We present a new method for solving the bi-objective multi-commodity minimum cost flow problem. This method is based on the standard bi-objective simplex method and Dantzig-Wolfe decomposition. The method is initialized by optimising the problem with respect to the first objective, a single objective multi-commodity flow problem, which is solved using standard Dantzig-Wolfe decomposition. Then, similar to the bi-objective simplex method, our method iteratively moves from one non-dominated extreme point to the next by finding entering variables with the maximum ratio of improvement of the second objective over deterioration of the first objective. As we use the Dantzig-Wolfe method in the initial iteration, we do not have a complete set of variables. Our method generates entering variables by finding the optimal solution of single commodity flow sub-problems with the ratio objective function. We find the optimal solution of each sub-problem among the efficient solutions of a bi-objective network flow problem. The solution with the best ratio objective value out of all sub-problems represents the entering variable. The method stops when all the non-dominated extreme points are obtained. The implementation of the method and numerical results are explained.

Key words: Linear bi-objective multi-commodity minimum cost flow problem, Dantzig-Wolfe decomposition, bi-objective simplex method

1 Introduction

The multi-commodity minimum cost flow problem ($MCMCF$) can be defined as a network optimisation problem where we want to send several commodities from their source nodes to their sink nodes. Individual commodities share arcs and compete for the capacity of the arcs. The *MCMCF* problem can be modelled as a linear optimisation problem with two sets of constraints: The flow conservation constraints and the capacity constraints which tie the commodities together. These constraints have a special block diagonal shape. Taking advantage of this special structure several decomposition approaches for solving the problem have been developed (see (Assad 1978) and references therein). In many application contexts of network models, there is more than one objective that has to be taken into account. Thus, multi-objective multi-commodity flow models may be more appropriate for modelling real-world decision making situations than the single objective models.

Let $G = (\mathcal{V}, \mathcal{A})$ be a directed network with a set of nodes or vertices $\mathcal{V} = \{1, 2, \ldots, n\}$ and a set of arcs $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V}$ with $|\mathcal{A}| = m$. Furthermore, let $\big((c_a^1)^k, (c_a^2)^k\big)$ be the pair of unit flow costs on arc $a \in \mathcal{A}$ for commodity k and x_a^k represent the amount of flow of commodity k going through arc $a \in \mathcal{A}$. Let $\mathbf{E}$ be the node arc incidence matrix of the network and let $\mathbf{x}^k$ be the flow vector (x_a^k, $a \in \mathcal{A}$). Let $\mathbf{b}^k$ be the balance vector for each commodity and $\mathbf{u}$ be the vector of arc capacities. By defining cost vectors $(\mathbf{c}^1)^k = ((c_a^1)^k, a \in \mathcal{A})$ and $(\mathbf{c}^2)^k = ((c_a^2)^k, a \in \mathcal{A})$. The bi-objective multi-commodity minimum cost flow problem (*BMCMCF*) can be written as the following bi-objective linear program

$$
\begin{array}{llllll}
\min & z(\mathbf{x}) = \left\{ \begin{array}{l} z_1(\mathbf{x}) = \sum_{k=1}^{q} (\mathbf{c}^1)^k \mathbf{x}^k \\ z_2(\mathbf{x}) = \sum_{k=1}^{q} (\mathbf{c}^2)^k \mathbf{x}^k \end{array} \right. & & & & \\
\text{s.t.} & \mathbf{E}\mathbf{x}^1 & & & & = \mathbf{b}^1 \\
& & \mathbf{E}\mathbf{x}^2 & & & = \mathbf{b}^2 \\
& & & & & \cdots \\
& & & \mathbf{E}\mathbf{x}^q & & = \mathbf{b}^q \\
& \mathbf{I}\mathbf{x}^1 + & \mathbf{I}\mathbf{x}^2 + \quad \cdots + & \mathbf{I}\mathbf{x}^q + & \mathbf{I}\mathbf{s} & = \mathbf{u} \\
& \mathbf{x}^k \geqq \mathbf{0}, & \text{for all } k = 1, 2, \ldots, q, & & &
\end{array} \tag{1}
$$

where I is the identity matrix and $\mathbf{s}$ is the vector of slack variables. The *BMCMCF* problem (1) is a bi-objective linear optimisation problem which can be solved by existing bi-objective linear programming algorithms (Moradi, Raith, and Ehrgott 2012). The bi-objective simplex method is one of the well known methods which can be applied here. The specially structured block diagonal constraint matrix of problem (1) permits the application of the Dantzig-Wolfe decomposition method. By integrating the bi-objective simplex method with the Dantzig-Wolfe decomposition method we present a new method for solving the *BMCMCF* problem which will be called bi-objective simplex decomposition (*BOSD*) method.

2 Background

Consider a bi-objective linear optimisation problem (*BLP*)

$$
\begin{array}{ll}
\min & \begin{pmatrix} z_1(\mathbf{x}) = (\mathbf{c}^1)^T \mathbf{x} \\ z_2(\mathbf{x}) = (\mathbf{c}^2)^T \mathbf{x} \end{pmatrix} \\
\text{s.t} & \mathbf{A}\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geqq \mathbf{0}.
\end{array}
$$

Definition 1. *Let $\mathcal{X}$ denote the feasible set of the BLP (2) and let $\mathcal{Z} = \{(z_1(\mathbf{x}), z_2(\mathbf{x})) : \mathbf{x} \in \mathcal{X}\}$ be the image of $\mathcal{X}$ under the objective functions. A feasible solution $\hat{x} \in \mathcal{X}$ of the BLP (2) is efficient if and only if there does not exist any $x' \in \mathcal{X}$ with $(z_1(\mathbf{x}'), z_2(\mathbf{x}')) \leqq (z_1(\hat{\mathbf{x}}), z_2(\hat{\mathbf{x}}))$ and $z(\mathbf{x}') \neq z(\hat{\mathbf{x}})$. The image of an efficient solution $z(\hat{\mathbf{x}}) = (z_1(\hat{\mathbf{x}}), z_2(\hat{\mathbf{x}}))$ is called non-dominated.*

Definition 2. *Let $f : \mathcal{X} \to \mathbb{R}$ be a function, where $\mathcal{X} \subseteq \mathbb{R}^n$ is a non-empty convex set. The function f is quasi-convex on $\mathcal{X}$ if*

$$f(\lambda x^1 + (1-\lambda)x^2) \leqq \max[f(x^1), f(x^2)]$$

for all x^1, $x^2 \in \mathcal{X}$ and for all $\lambda \in [0,1]$. Function f is explicitly quasi-convex if it is quasi-convex and if $f(x^1) \neq f(x^2)$ implies

$$f(\lambda x^1 + (1-\lambda)x^2) < \max[f(x^1), f(x^2)]$$

for all $\lambda \in (0,1)$. The function f is quasi-concave (explicitly quasi-concave) on $\mathcal{X}$ if $(-f)$ is quasi-convex (explicitly quasi-convex) on $\mathcal{X}$.

2.1 Bi-objective simplex method

Modelling the *BMCMCF* problem as a liner program permits the use of the standard bi-objective simplex method. Ehrgott (2005) gives a comprehensive explanation of this method and we use the same notation here. The bi-objective simplex method initially starts by optimising the problem with respect to the first objective. The method then iteratively moves from one non-dominated extreme point to the next by finding entering variables with the maximum ratio of improvement of the second objective over the deterioration of the first objective. The method stops when all the non-dominated extreme points are obtained. The initial solution may be weakly efficient and the algorithm may find some solutions that do not define non-dominated extreme points. These solutions can be easily discarded at the end of the algorithm.

2.2 Dantzig-Wolfe decomposition method for the *MCMCF* problem

Tomlin (1966) has devised an algorithm for solving the *MCMCF* problem based on the Dantzig-Wolfe decomposition method. We follow the same notation in this paper. In this method the original problem is reformulated into a master problem over a set of bundle (complicating) constraints and k shortest path sub-problems, each over a set of network balance constraints. Starting with a basic feasible solution, the master problem iteratively updates a set of cost coefficients for the sub-problems. Based on these cost coefficients the optimal solutions of each sub-problem are obtained. These solutions are the most improving columns (corresponding to non-basic variables) to enter the basis for each commodity. This method continues until no sub-problem can find a column to improve the master problem.

Let $\mathcal{Y}_k = \{\mathbf{y}_1^k, \mathbf{y}_2^k, ..., \mathbf{y}_{n_k}^k\}$ be the extreme points of $\mathcal{X}_k = \{\mathbf{x}^k : \mathbf{E}\mathbf{x}^k = \mathbf{b}^k\}$ for $k = 1, 2, ...q$. Then any $\mathbf{x}^k$ can be expressed as a convex combination of the elements of $\mathcal{Y}_k$ as follows:

$$\mathbf{x}^k = \sum_{j=1}^{n_k} \lambda_j^k \mathbf{y}_j^k, \quad \sum_{j=1}^{n_k} \lambda_j^k = 1, \quad \lambda_j^k \geqq 0, \quad j = 1, 2, ..., n_k. \tag{2}$$

Substituting (2) for $\mathbf{x}^k$ in the *MCMCF* problem we get the following problem.

$$\min \quad z(\lambda) = \sum_{k=1}^{q}\sum_{j=1}^{n_k} \lambda_j^k((\mathbf{c}^1)^k \mathbf{y}_j^k)$$

$$\begin{array}{llllll|c}
 & & & & & & \text{dual} \\
\text{s.t.} & \sum_{j=1}^{n_1} \lambda_j^1 & & & & = \mathbf{1} & \alpha_1 \\
 & & \sum_{j=1}^{n_2} \lambda_j^2 & & & = \mathbf{1} & \alpha_2 \\
 & & & & & \cdots & \\
 & & & \sum_{j=1}^{n_q} \lambda_j^q & & = \mathbf{1} & \alpha_q \\
 & \sum_{j=1}^{n_1} \lambda_j^1(\mathbf{I}\mathbf{y}_j^1)+ & \sum_{j=1}^{n_2} \lambda_j^2(\mathbf{I}\mathbf{y}_j^2)+ \quad \cdots + & \sum_{j=1}^{n_q} \lambda_j^q(\mathbf{I}\mathbf{y}_j^q)+ & \mathbf{Is} & = \mathbf{u}. & \mathbf{w}
\end{array}$$

$$\lambda_j^k \geqq 0, \text{ for all } j = 1, 2, ..., n_k, k = 1, 2, \ldots, q. \tag{3}$$

Suppose that we have a basic feasible solution for problem (3), in terms of λ_j^k variables. Let $(\alpha, \mathbf{w})$ be the vector of dual variables for this basic feasible solution where α has q components and $\mathbf{w}$ has m components. For this solution to be optimal, it must also be feasible for the dual problem. This condition is called dual feasibility and for problem (3) can be stated as:

- $w_a \geq 0$ for any slack variable s_a, and
- $(\mathbf{c}^1)^k \mathbf{y}_j^k - \mathbf{w}\mathbf{y}_j^k - \alpha^k \geqq 0$ corresponding to each λ_j^k.

If the first condition is violated for any slack variable that slack variable will be introduced into the master basis. If the second condition is violated the corresponding λ_j^k variable is a candidate to enter the master basis. For the commodity k the most improving column j (non-basic λ_j^k variable) to enter the basis can be found by solving sub-problem

$$\begin{aligned}
\min \quad & g(\mathbf{y}^k) = ((\mathbf{c}^1)^k - \mathbf{w})^T \mathbf{y}^k - \alpha^k \\
\text{s.t.} \quad & \mathbf{E}\mathbf{y}^k = \mathbf{b}^k, \\
& \mathbf{y}^k \geqq \mathbf{0}.
\end{aligned} \tag{4}$$

If the optimal solution $\mathbf{y}_j^k$ of problem (4) has a negative objective value, non-basic λ_j^k variable will be introduced into the master basis. After finding the entering column, it is adjoined to the master basis and the leaving variable is determined in the usual way. By pivoting, the basis inverse, dual variables, and right-hand-side are updated. The method continues until there does not exist any candidate variable to enter the basis, which means the optimal solution of the *MCMCF* problem is obtained.

3 Bi-objective simplex decomposition method

In this section we present the *BOSD* method for solving the *BMCMCF* problem. Applying the change of variables (2), the *BMCMCF* problem (1) can be written as

$$\min z(\lambda) = \begin{cases} z_1(\lambda) = \displaystyle\sum_{k=1}^{q}\sum_{j=1}^{n_k} \lambda_j^k((\mathbf{c}^1)^k \mathbf{y}_j^k) \\ z_2(\lambda) = \displaystyle\sum_{k=1}^{q}\sum_{j=1}^{n_k} \lambda_j^k((\mathbf{c}^2)^k \mathbf{y}_j^k) \end{cases}$$

$$\begin{array}{lllllll|ll}
 & & & & & & & \text{dual} & \\
\text{s.t.} & \displaystyle\sum_{j=1}^{n_1}\lambda_j^1 & & & & & = \mathbf{1} & \alpha_1^1 & \alpha_1^2 \\
 & & \displaystyle\sum_{j=1}^{n_2}\lambda_j^2 & & & & = \mathbf{1} & \alpha_2^1 & \alpha_2^2 \\
 & & & & & & \dots & & \\
 & & & & \displaystyle\sum_{j=1}^{n_q}\lambda_j^q & & = \mathbf{1} & \alpha_q^1 & \alpha_q^2 \\
 & \displaystyle\sum_{j=1}^{n_1}\lambda_j^1(\mathrm{I}y_j^1)+ & \displaystyle\sum_{j=1}^{n_2}\lambda_j^2(\mathrm{I}y_j^2)+ & \cdots + & \displaystyle\sum_{j=1}^{n_q}\lambda_j^q(\mathrm{I}\mathbf{y}_j^q)+ & \mathrm{I}\mathbf{s} & = \mathbf{u}. & \mathbf{w}^1 & \mathbf{w}^2
\end{array}$$

$$\lambda_j^k \geqq 0, \quad \text{for all } j = 1, 2, ..., n_k, k = 1, 2, \dots, q. \tag{5}$$

The *BOSD* method initially starts by obtaining a solution which is minimal with respect to the first objective component. In this step problem (3) becomes a single objective *MCMCF* problem, and the standard Dantzig-Wolfe decomposition method, as explained in Section 2.2, can be applied here. Let $\mathbf{x}_i$ be the initial solution and let $\mathcal{B}$, $(\alpha^1, \mathbf{w}^1)$ and $(\alpha^2, \mathbf{w}^2)$, respectively, be the corresponding master basis and the vectors of dual variables for the first and second objective. The *BOSD* method then iteratively continues by finding an entering variable which has the maximal ratio of improvement of the second objective function over deterioration of the first. Slack variable s_a $(a \in \mathcal{A})$ is a candidate for introduction to the master basis if

$$w_a^1 < 0 \text{ and } w_a^2 \geqq 0.$$

Non-basic variable λ_j^k is a candidate for introduction to the master basis if

$$(\mathbf{c}^2 - (w_j^2)^k)^T \mathbf{y}_j^k - (\alpha^2)^k < 0$$
$$(\mathbf{c}^1 - (w_j^1)^k)^T \mathbf{y}_j^k - (\alpha^1)^k \geqq 0.$$

The ratio for slack variable s_a can be easily obtained from

$$\mu_a = \frac{-w_a^2}{w_a^1 - w_a^2}. \tag{6}$$

For commodity k the most improving column j (variable λ_j^k) can be found by solving the following fractional optimisation sub-problem:

$$\max \quad g(\mathbf{y}^k) = \frac{-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}^k-(\alpha^2)^k)}{(\mathbf{c}^1-\mathbf{w}^1)^T\mathbf{y}^k-(\alpha^1)^k-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}^k-(\alpha^2)^k)}$$
$$\begin{aligned} \text{s.t} \quad & \mathbf{E}\mathbf{y}^k = \mathbf{b}^k, \\ & (\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}^k-(\alpha^2)^k < \mathbf{0}, \\ & (\mathbf{c}^1-\mathbf{w}^1)^T\mathbf{y}^k-(\alpha^1)^k \geqq \mathbf{0}, \\ & \mathbf{y}^k \geqq \mathbf{0}. \end{aligned} \tag{7}$$

If the optimal solution of problem (7) $\mathbf{y}_j^k$ has a positive objective value, non-basic λ_j^k variable is a candidate to enter the basis. Among the entering variable candidates we choose the one with the maximum ratio. The identified entering variable is adjoined to the master basis and the leaving variable is determined according to standard simplex rules. The *BOSD* methods continues until there does not exist any entering variable which can improve the second objective by deteriorating the first objective, which implies that all non-dominated extreme points are obtained. The *BOSD* Algorithm is stated as Algorithm 3.

Algorithm 1 (Bi-Objective Simplex Decomposition Algorithm)

1: ***Input:*** Data $\mathbf{E}$, $\mathbf{b}$, $\mathbf{c}^1$, and $\mathbf{c}^2$ for a *BMCMCF* problem.
2: Obtain an optimal solution and an optimal master basis $\mathcal{B}$ with respect to the first objective by standard Dantzig-Wolfe decomposition. Compute $\bar{\mathbf{b}} = \mathcal{B}^{-1}\begin{pmatrix}\mathbf{1}\\ \mathbf{u}\end{pmatrix}$, $(\alpha^1, \mathbf{w}^1) = \hat{c}^1_{\mathcal{B}}\mathcal{B}^{-1}$, $(\alpha^2, \mathbf{w}^2) = \hat{c}^2_{\mathcal{B}}\mathcal{B}^{-1}$, where $\hat{c}_j^{1k} = (\mathbf{c}^1)^k\mathbf{y}_j^k$ and $\hat{c}_j^{2k} = (\mathbf{c}^2)^k\mathbf{y}_j^k$ for λ_j^k variables, α^1 and α^2 have q components and $\mathbf{w}^1$ and $\mathbf{w}^2$ have m components.
3: Let $\mathcal{T} = \emptyset$ and $\mathcal{I} = \{a \in \{1, 2, \ldots, m\} : w_a^2 < 0,\ w_a^1 \geq 0\}$.
4: **if** $\mathcal{I} \neq \emptyset$ **then**
5: $\quad t_1 \in \text{argmax}\{i \in \mathcal{I} : \mu_i = \frac{-w_i^2}{w_i^1 - w_i^2}\}$, $\mathcal{T} = \mathcal{T} \cup \{s_{t_1}\}$.
6: **end if**
7: For each commodity k solve fractional optimisation problem (7) and find the optimal solution $\mathbf{y}_j^k$.
8: $t_2 \in \text{argmax}\{k \in \{1, 2, \ldots, q\}\ : g(\mathbf{y}_j^k)\ > 0\}$, $\mathcal{T} = \mathcal{T} \cup \{\lambda_j^{t_2}\}$.
9: **if** $\mathcal{T} \neq \emptyset$ **then**
10: $\quad$ Among the variables in $\mathcal{T}$ choose t with the maximum ratio.
11: $\quad$ Identify $r \in \text{argmin}\{l \in \mathcal{B} : \frac{\bar{b}_l}{\tilde{\mathbf{E}}_{tl}}, \tilde{\mathbf{E}}_{tl} > 0\}$, where $\tilde{\mathbf{E}}_t$, $t \in \mathcal{T}$, is an entering column.
12: $\quad$ Let $\mathcal{B} := (\mathcal{B} \setminus \{r\}) \cup \{t\}$ and update $\bar{\mathbf{b}}$, $(\alpha^1, \mathbf{w}^1)$ and $(\alpha^2, \mathbf{w}^2)$.
13: $\quad$ **return** to Step 3.
14: **end if**
15: ***Output:*** All non-dominated extreme points

4 Solving the fractional sub-problem

We establish a method to find the optimal solution of sub-problem (7) among the efficient solutions of a bi-objective linear optimisation problem.

Theorem 1. *If $\mathbf{y}_o^k$ is an optimal solution of problem (7) it is among the efficient solutions of following bi-objective linear optimisation problem.*

$$\min \begin{pmatrix} (\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}^k-(\alpha^2)^k \\ (\mathbf{c}^1-\mathbf{w}^1)^T\mathbf{y}^k-(\alpha^1)^k-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}^k-(\alpha^2)^k)) \end{pmatrix} \tag{8}$$
$$s.t. \quad \mathbf{E}\mathbf{y}^k=\mathbf{b}^k,$$
$$\mathbf{y^k} \geq \mathbf{0}.$$

Proof. Let $\mathbf{y}_o^k$ be the optimal solution of problem (7). If it is not an efficient solution of problem (8), then there exists an efficient solution $\mathbf{y}_e^k$ of problem (8) with

$$(\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_o^k \leqq (\mathbf{c^2}-\mathbf{w^2})^T\mathbf{y}_e^k \text{ and} \tag{9}$$
$$(\mathbf{c}^1-\mathbf{w}^1)^T\mathbf{y}_o^k-(\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_o^k \leqq (\mathbf{c}^1-\mathbf{w}^1)^T\mathbf{y}_e^k-(\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_e^k. \tag{10}$$

At least one of the inequalities (9) and (10) is a strict inequality. Since $\mathbf{y}_o^k$ is an optimal solution of problem (7) we have $(\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_o^k-(\alpha^2)^k<0$ and from equation (9) we have $(\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_e^k-(\alpha^2)^k<0$. But $\mathbf{y}_e^k$ is an efficient solution which means that the reduced cost cannot be negative in both components for this solution which means $(\mathbf{c}^1-\mathbf{w}^1)^T\mathbf{y}_e^k-(\alpha^1)^k \geqq \mathbf{0}$ and therefore

$$(\mathbf{c}^1-\mathbf{w}^1)^T\mathbf{y}_e^k-(\alpha^1)^k-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_e^k-(\alpha^2)^k)>0 \text{ and} \tag{11}$$
$$(\mathbf{c}^1-\mathbf{w}^1)^T\mathbf{y}_o^k-(\alpha^1)^k-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_o^k-(\alpha^2)^k)>0. \tag{12}$$

From equations (9) and (11) it follows that

$$\frac{-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_e^k-(\alpha^2)^k)}{(\mathbf{c}^1-\mathbf{w}^1)^T\mathbf{y}_e^k-(\alpha^1)^k-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_e^k-(\alpha^2)^k)} \geqq \frac{-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_o^k-(\alpha^2)^k)}{(\mathbf{c}^1-\mathbf{w}^1)^T\mathbf{y}_e^k-(\alpha^1)^k-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_e^k-(\alpha^2)^k)} \tag{13}$$

From equations (10) and (12) it follows that

$$\frac{-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_o^k-(\alpha^2)^k)}{(\mathbf{c}^1-\mathbf{w}^1)^T\mathbf{y}_e^k-(\alpha^1)^k-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_e^k-(\alpha^2)^k)} \geqq \frac{-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_o^k-(\alpha^2)^k)}{(\mathbf{c}^1-\mathbf{w}^1)^T\mathbf{y}_o^k-(\alpha^1)^k-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_o^k-(\alpha^2)^k)} \tag{14}$$

At least one of the inequalities (13) and (14) is strict, and it follows that

$$\frac{-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_e^k-(\alpha^2)^k))}{(\mathbf{c}^1-\mathbf{w}^1)^T\mathbf{y}_e^k(\alpha^1)^k-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_e^k(\alpha^2)^k)} > \frac{-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_o^k-(\alpha^2)^k)}{(\mathbf{c}^1-\mathbf{w}^1)^T\mathbf{y}_o^k-(\alpha^1)^k-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}_o^k-(\alpha^1)^k)}. \tag{15}$$

Which contradicts the maximality of $\mathbf{y}_o^k$ for sub-problem (7). □

Theorem 2. *Any locally maximal solution of problem (7) is a global maximum. Also, the problem reaches its global maximum in one or more extreme point of the feasible set.*

Proof. $\frac{-(\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}^k-(\alpha^2)^k}{(\mathbf{c}^1-\mathbf{w}^1)^T\mathbf{y}^k-(\alpha^1)^k-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}^k-(\alpha^2)^k)}$ is a linear fractional function therefore, it is both explicitly quasi-convex and explicitly quasi-concave. For explicit quasi-convex functions on compact and convex sets any local maximum is a global maximum. Explicit quasi-convex functions on compact and convex sets reach their global maximum in one or more extreme point of the feasible set (Stancu-Minasian 1997). □

Theorems 1 and 2 imply that the optimal solution of sub-problem (7) can be obtained through the following steps. Initially problem (8) is solved with respect to one of the objectives. The algorithm then moves iteratively from one extreme point of problem (8) to the next until it reached the local optimum for objective function $\frac{-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}^k-(\alpha^2)^k)}{(\mathbf{c}^1-\mathbf{w}^1)^T\mathbf{y}^k-(\alpha^1)^k-((\mathbf{c}^2-\mathbf{w}^2)^T\mathbf{y}^k-(\alpha^2)^k)}$.

5 Computational results

In this section, we discuss the implementation of the method. We provide computational results obtained from several types of directed bi-objective network instances with two, three or five commodities. We used three groups of network examples with the same structure used by (Raith and Ehrgott 2009b). We modified their bi-objective single-commodity instances to include several commodities. The first two groups are directed network instances generated by the NETGEN (Klingman, Napier, and Stutz 1974) generator which is modified to include a second objective function and multiple commodities. Table 1 shows NETGEN parameters for the generation of each set of networks, such as number of nodes, arcs, sources and sinks, etc. Problems N01 – N12 have varying sum of supply for each commodity ($\sum_{i\in\mathcal{V}:b_i>0} b_i^k$) and problems F01 – F12 have fixed sum of supply for each commodity. There are 30 problems for each set of parameters. The third group of network instances consists of networks with a grid structure. In these networks, nodes are arranged in a rectangular grid with given parameters height h, width w, maximum cost c_{max}, maximum capacity $u-max$ and sum of supply ($\sum_{i\in\mathcal{V}:b_i>0} b_i^k$). All grid instances are listed in Table 2.

In Tables 3 – 5, the average number of non-dominated extreme points $\mid \mathcal{Y}_{ex} \mid$ as well as the average CPU time $t_{avg}(s)$ for different numbers of commodities are presented. From these tables the following observations can be made:

- Our method solves all of these instances in reasonable time. Average CPU times are between 0.70 and 238.27 seconds.

Table 1: NETGEN test instances.

Name	Nodes	Arcs	Sources	Sinks	$\sum_{i\in\mathcal{V}:b_i^k>0} b_i^k$	Transshipment sources	Transshipment sinks
N01/F01	20	60	9	7	90/100	4	3
N02/F02	20	80	9	7	90/100	4	3
N03/F03	20	100	9	7	90/100	4	3
N04/F04	40	120	18	14	180/100	9	7
N05/F05	40	160	18	14	180/100	9	7
N06/F06	40	200	18	14	180/100	9	7
N07/F07	60	180	27	21	270/100	14	10
N08/F08	60	240	27	21	270/100	14	10
N09/F09	60	300	27	21	270/100	14	10
N10/F10	80	240	35	38	350/100	17	14
N11/F11	80	320	35	38	350/100	17	14
N12/F12	80	400	35	38	350/100	17	14

- It can be seen that by increasing the number of commodities the CPU running-time increases significantly. This happens because the size of problem and the number of variables are proportional to the number of commodities.

- The number of non-dominated extreme points $\mid \mathcal{Y}_{ex} \mid$ and consequently the average CPU time increases by increasing the number of nodes or by increasing the number of arcs.

6 Future work

In future research we will address the application of other methods for solving the sub-problem (7), such as the Charnes-Cooper transformation method (Charnes and Cooper 1962) and the extended Simplex algorithm for fractional problems (Swarup 1965). Furthermore, we will investigate the use of bi-objective shortest path methods (Raith and Ehrgott 2009a) for solving the bi-objective linear problem (8).

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Table 2: Grid test instances.

Name	h	w	Nodes	Arcs	c_{max}	u_{max}	$\sum_{i \in \mathcal{V}: b_i^k > 0} b_i^k$
G01	4	5	20	62	100	50	100
G02	5	8	40	134	100	50	100
G03	6	10	60	208	100	50	100
G04	8	10	80	284	100	50	100
G05	6	10	60	208	100	75	100
G06	6	10	60	208	100	100	100
G07	6	10	60	208	25	50	100
G08	6	10	60	208	50	50	100
G09	8	10	80	284	100	75	100
G10	8	10	80	284	100	100	100
G11	8	10	80	284	25	50	100
G12	8	10	80	284	50	50	100

Table 3: Results for NETGEN instances with varying total supply.

	2-commodity		**3-commodity**		**5-commodity**	
Name	$\mid \mathcal{Y}_{ex} \mid$	$t_{avg}(s)$	$\mid \mathcal{Y}_{ex} \mid$	$t_{avg}(s)$	$\mid \mathcal{Y}_{ex} \mid$	$t_{avg}(s)$
N01	25.17	1.74	35.90	2.44	56.43	5.29
N02	32.47	2.07	45.43	2.92	72.97	6.46
N03	36.23	2.06	52.20	3.33	85.60	7.79
N04	53.30	13.23	71.90	12.64	119.80	25.03
N05	68.27	14.52	100.97	17.45	158.80	33.68
N06	80.40	16.29	116.87	20.40	189.27	41.48
N07	75.80	48.00	111.38	48.72	179.03	74.26
N08	104.37	45.63	155.00	52.59	246.40	97.91
N09	132.47	69.90	182.30	58.54	299.00	115.63
N10	106.63	112.09	152.87	107.11	242.03	144.60
N11	140.50	137.80	206.20	118.83	331.27	188.46
N12	169.37	168.54	248.33	148.13	398.03	238.27

Table 4: Results for NETGEN instances with fixed total supply.

	2-commodity		**3-commodity**		**5-commodity**	
Name	$\mid \mathcal{Y}_{ex} \mid$	$t_{avg}(s)$	$\mid \mathcal{Y}_{ex} \mid$	$t_{avg}(s)$	$\mid \mathcal{Y}_{ex} \mid$	$t_{avg}(s)$
F01	25.43	1.75	36.40	2.50	58.63	5.10
F02	34.10	2.08	46.97	2.98	74.47	6.49
F03	38.73	2.24	55.27	3.43	89.43	8.12
F04	43.20	8.53	62.69	10.34	102.38	20.77
F05	58.57	12.25	84.87	13.22	139.83	28.59
F06	65.57	9.95	97.50	13.93	160.87	35.38
F07	46.60	24.23	71.55	31.91	121.97	44.97
F08	71.67	30.64	105.73	33.75	177.60	63.07
F09	83.20	32.70	124.50	35.46	204.43	75.65
F10	46.07	55.04	71.87	64.11	120.39	78.95
F11	67.47	55.22	103.07	82.91	181.27	107.56
F12	86.27	75.60	132.70	69.02	220.03	131.67

Table 5: Results for Grid instances.

	2-commodity		**3-commodity**		**5-commodity**	
Name	$\mid \mathcal{Y}_{ex} \mid$	$t_{avg}(s)$	$\mid \mathcal{Y}_{ex} \mid$	$t_{avg}(s)$	$\mid \mathcal{Y}_{ex} \mid$	$t_{avg}(s)$
G01	13.23	0.70	19.73	1.12	30.47	2.51
G02	26.87	3.78	37.40	5.03	59.77	10.50
G03	36.73	8.49	52.83	11.59	86.63	26.51
G04	45.90	15.11	67.10	23.24	113.17	52.46
G05	36.90	7.99	52.67	10.75	86.87	26.37
G06	36.03	5.53	52.47	10.13	87.57	26.69
G07	33.27	8.31	46.47	10.17	72.93	23.23
G08	35.63	8.85	52.67	11.93	83.63	25.05
G09	45.40	13.61	67.00	20.46	110.50	51.76
G10	44.70	10.27	67.23	20.32	110.53	52.14
G11	41.93	15.54	57.40	18.93	91.53	45.64
G12	44.07	14.45	63.43	19.45	104.23	48.41