Diamond Cutting Problem for Non-Convex and Multiple Diamonds

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Abstract

The diamond cutting problem involves maximising the size of a given diamond shape required to lie within a given rough stone. The diamond can be rotated, translated and magnified to find the optimal orientation. We extended the diamond cutting problem to incorporate non-convex stones and multiple diamonds in two dimensions. These additions are important for realistic modelling, because the majority of rough diamond stones have dents or imperfections which make them non-convex. Furthermore, the combined value of several diamonds cut from a stone is often significantly greater than the value of one large diamond. However the modelling is complex because even for a single diamond within a convex stone, the diamond cutting method is non-linear and non-convex.

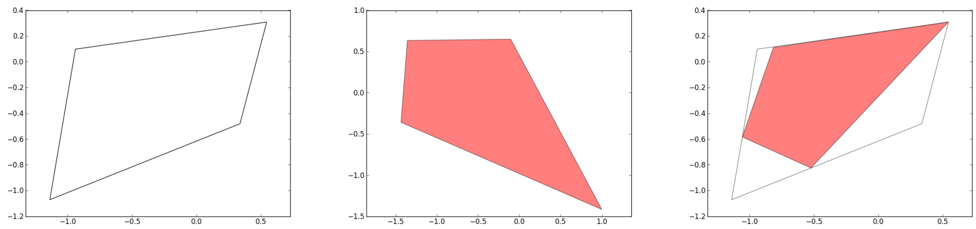
We developed an iterative method to solve the diamond cutting problem for non-convex stones. This formulation has linear constraints and a non-linear objective. Although not guaranteed to produce the global optimum solution, we can identify a small range bounding the global optimum, related to the accuracy of a piecewise linear approximation to the objective function. This method can be extended to place multiple diamonds using either a greedy sequential algorithm or a more complex simultaneous placement method.

1 Background

1.1 Single diamond in a convex stone

Given an inner polygon and an outer polygon, the *diamond cutting problem* involves transforming the inner polygon to maximise its size, while remaining fully enclosed by the outer polygon. The transformation can include a translation, a rotation and a magnification. We will refer to the inner polygon as the diamond, and the outer polygon as the rough stone, or just stone. An example solution for an arbitrary diamond and stone is shown in figure 1.

To simplify the problem, we focus on solving the diamond cutting problem in two dimensions. This allows the model to be extended to include non-convex stones



(a) Stone. **(b)** Diamond. **(c)** Optimal fit.

Figure 1: Example solution for the diamond cutting problem.

and multiple diamonds. The same underlying formulation can be applied to the three dimensional problem, but solving this problem is non-trivial and is an area for further investigation.

1.2 Related problems

There are a range of problems that involve fitting one or more non-overlapping polygons inside another polygonal container. This process is relevant to several manufacturing industries including shipbuilding, apparel, sheet metal and leather products (Hooper 1985). Specific problems studied in the literature include the Containment problem first solved by Chazelle (1985) and the Minimum Enclosure problem solved by Milenkovic (1999). The Containment problem is concerned with placing polygons that can not magnify, while the Minimum Enclosure problem typically minimises the area of a rectangular outer polygon. Both these problems are more restrictive than the diamond cutting problem, so the methodologies outlined in this paper may be applied to the manufacturing industry with further development.

1.3 Formulation

We will first show how the diamond cutting problem for a single diamond within a convex stone can be modelled as a mathematical program, as first proposed by Viswambharan (2002).

Indices

V = the set of vertices of the diamond;
E = the set of edges of the stone.

Parameters

x_i, y_i = coordinates of the i^{th} vertex of the diamond;
a_j, b_j, c_j = coefficients of the hyperplane representing the j^{th} edge of the rough stone.

Decision variables

s = scale the diamond is magnified by;
x_{off}, y_{off} = translation of the diamond;
θ = angle of rotation of the diamond;
$\tilde{x}, \tilde{y}$ = transformed diamond vertices.

Model Single Convex Polygon

$$\max s \tag{1}$$
$$\text{s.t. } a_j\tilde{x}_i + b_j\tilde{y}_i \leq c_j \qquad \forall i \in V, \quad \forall j \in E \tag{2}$$
$$\tilde{x}_i = s(x_i \cos\theta - y_i \sin\theta) + x_{off} \qquad \forall i \in V \tag{3}$$
$$\tilde{y}_i = s(x_i \sin\theta + y_i \cos\theta) + y_{off} \qquad \forall i \in V \tag{4}$$
$$0 \leq \theta \leq 2\pi \tag{5}$$
$$s \geq 0, \quad x_{off}, y_{off}, \tilde{x}_i, \tilde{y}_i \text{ free} \tag{6}$$

Constraints (3) and (4) describe how the diamond is transformed. We are given an initial polygonal diamond with coordinates (x_i, y_i) which we transform to find the best fit within the stone. This transformation comprises a magnification (s), a translation (x_{off}, y_{off}) and a rotation by an angle (θ). The rotation involves multiplying the original diamond coordinates by a rotation matrix.

Constraint (2) requires the transformed diamond to be fully enclosed by the stone. We define the stone mathematically as the intersection of a set of half-spaces as shown in figure 2. Each edge of the stone can be represented as a half-space of the form of the form $ax + by \leq c$. Every transformed diamond vertex $(\tilde{x}, \tilde{y})$ is required to lie within the intersection of these half-spaces.

The objective is to simply maximise the magnification s of the transformed diamond.

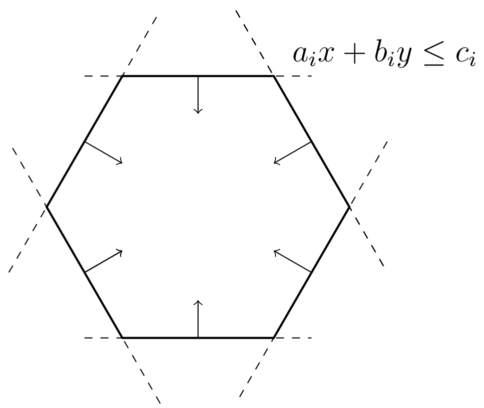


Figure 2: The intersection of a set of half-spaces defining a hexagonal stone.

This problem is difficult to solve because constraints (3) and (4) are non-linear. Furthermore, the objective function is non-convex with respect to the angle of rotation (θ) of the diamond as shown in figure 3.

2 Weighted average method

As first proposed by Fatt (2012), the problem can be reformulated using an alternative method to rotate the diamond. We take a convex combination of two stationary diamonds separated by a fixed angle of rotation. Depending on the weighting between these two stationary diamonds, it is possible to form an intermediate rotated diamond (shaded in figure 4) at any angle between them.

This reformulation is an improvement over the previous formulation because the reformulated constraints are linear. Although the reformulated objective function is

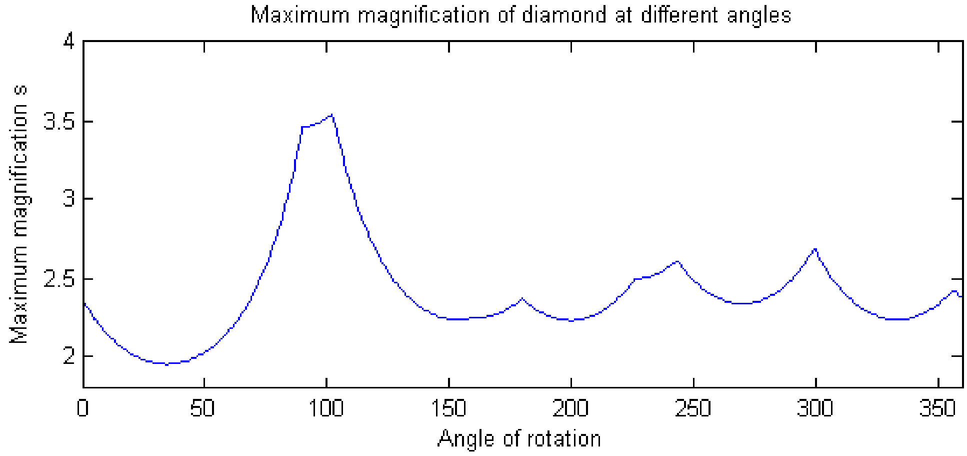
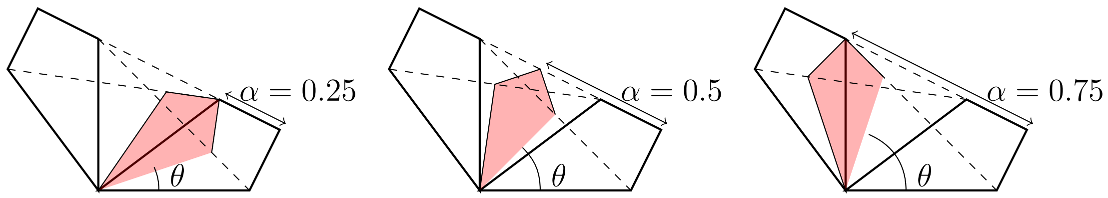


Figure 3: How magnification varies with angle of rotation (θ).

non-linear, it is easier to approximate than the original non-linear constraints. The reformulated non-linear objective can be linearised with a piecewise linear function and solved as an integer program. While this approximation is not guaranteed to give the global optimum, if we approximate the non-linear constraints from the original formulation, then our approximate solution may not even be feasible.



(a) $\theta = 0.32$ rad **(b)** $\theta = 0.79$ rad **(c)** $\theta = 1.25$ rad

Figure 4: Shaded diamonds with different angles of rotation can be formed by changing the parameter α.

2.1 Transformation of diamond

Suppose we have two identical diamonds, separated by an angle of rotation ϕ around the origin. These diamonds are unshaded in figure 4, and have coordinates (x_i, y_i) and $(\hat{x}_i, \hat{y}_i)$ respectively. The vertices of these diamonds can be represented using a polar co-ordinate system with corresponding θ_i and r_i values:

$$x_i = r_i \cos\theta_i, \quad y_i = r_i \sin\theta_i, \qquad \hat{x}_i = r_i \cos(\theta_i + \phi), \quad \hat{y}_i = r_i \sin(\theta_i + \phi) \tag{7}$$

The shaded diamond $(\tilde{x}_i, \tilde{y}_i)$, can be formed as a weighted average of the two unshaded diamonds. Any diamond can be formed at an angle between 0 and ϕ by choosing an appropriate weight α to control its position. In this formulation α is a proxy for the rotation θ of the diamond. It can be shown using simple trigonometry that $\theta = \arctan \frac{\alpha}{1-\alpha}$, when $\phi = \frac{\pi}{2}$.

$$\tilde{x}_i = \alpha x_i + (1-\alpha)\hat{x}_i \qquad \tilde{y}_i = \alpha y_i + (1-\alpha)\hat{y}_i \tag{8}$$

In the original formulation, the diamond was transformed by being translated, rotated and magnified. If we include translation x_{off}, y_{off} and a magnification $\hat{s}$, then the transformed diamond vertices can be given by

$$\tilde{x}_i = \hat{s}(\alpha x_i + (1-\alpha)\hat{x}_i) + x_{off}, \quad \tilde{y}_i = \hat{s}(\alpha y_i + (1-\alpha)\hat{y}_i) + y_{off} \tag{9}$$

We can make this expression linear by introducing a new variable $\beta = \alpha\hat{s}$, which does not have a physical interpretation but is convenient for modelling.

$$\tilde{x} = \beta x_i + (\hat{s}-\beta)\hat{x}_i + x_{off}, \qquad \tilde{y} = \beta y_i + (\hat{s}-\beta)\hat{y}_i + y_{off} \tag{10}$$

2.2 Diamond enclosure constraint

As in the original formulation, the diamond vertices are required to lie within the intersection of the set of half-spaces that define the stone.

We have shown it is possible to form any transformed diamond rotated to an angle θ in the range $[0, \phi]$, where ϕ is the angle separating the two stationary diamonds. To achieve a full 2π radians of rotation, we need to solve multiple linear problems. For example, if $\phi = \frac{\pi}{2}$ (as in figure 4), then the problem needs to be solved four times to find the maximum magnification in each quadrant $[0, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \pi]$, $[\pi, \frac{3\pi}{2}]$, $[\frac{3\pi}{2}, 2\pi]$.

Alternatively, the problem can be condensed to a single integer problem by using the big-M method. Suppose there are 4 sets of fixed diamond co-ordinates indexed $k \in K$, each rotated $\frac{\pi}{2}$ from the previous set. The binary variables Z_k controls which quadrant the diamond is in, or which set of constraints is active, leading to the following constraints:

$$a_j\tilde{x}_i + b_j\tilde{y}_i - MZ_k \le c_j \qquad \forall i \in V, \forall j \in E, \forall k \in K \tag{11}$$

$$\sum_{k=1}^{4} Z_k = 3. \tag{12}$$

2.3 Objective function

As observed in figure 4, the shaded diamond is smaller than the original unshaded diamonds. The scale of the original unshaded and transformed shaded diamonds can be compared through a factor f. This is found by comparing the radial components of corresponding vertices of the original and transformed diamonds, r_i and $\tilde{r}_i$ respectively:

$$f = \frac{\tilde{r}_i}{r_i} = \sqrt{\alpha^2 + (1-\alpha)^2 + 2(1-\alpha)\alpha\cos\phi} \tag{13}$$

Therefore the transformed objective function is

$$s = \hat{s}f = \hat{s}\sqrt{\alpha^2 + (1-\alpha)^2 + 2(1-\alpha)\alpha\cos\phi} \tag{14}$$

$$= \sqrt{\beta^2 + (\hat{s}-\beta)^2 + 2\beta(\hat{s}-\beta)\cos\phi} \tag{15}$$

By selecting $\phi = \frac{\pi}{2}$ this can be reduced to $\sqrt{\beta^2 + (\hat{s}-\beta)^2}$. Furthermore, because the square-root function is monotonic, we can simplify the objective to $\beta^2 + (\hat{s}-\beta)^2$ and get the same optimal solution. Each quadratic term can be modelled using a piecewise linear function.

The objective value z found by solving this approximation may not be the global optimal solution z^*, but we can find bounds the global optimal solution will lie within. As proven in Biggs (2013), for a outer piecewise linear approximation with uniform width b between breakpoints, the bounds can be shown to be

$$z - \frac{b^2}{2} \leq z^* \leq z. \tag{16}$$

Therefore the full formulation is

$$\begin{aligned}
\max \quad & \beta^2 + (\hat{s} - \beta)^2 && && (17)\\
\text{s.t.} \quad & a_j\tilde{x}_i + b_j\tilde{y}_i - MZ_k \leq c_j && \forall i \in V, \quad \forall j \in E, \quad \forall k \in K && (18)\\
& \tilde{x}_i = \beta x_i + (\hat{s} - \beta)\hat{x}_i + x_{off} && \forall i \in V && (19)\\
& \tilde{y}_i = \beta y_i + (\hat{s} - \beta)\hat{y}_i + y_{off} && \forall i \in V && (20)\\
& \sum_{k=1}^{4} Z_k = 3 && && (21)\\
& 0 \leq \beta \leq \hat{s} && && (22)\\
& \hat{s} \geq 0; \quad x_{off}, y_{off}, \tilde{x}_i, \tilde{y}_i \quad \text{free.} && && (23)
\end{aligned}$$

This was coded in Python and solved using Gurobi.

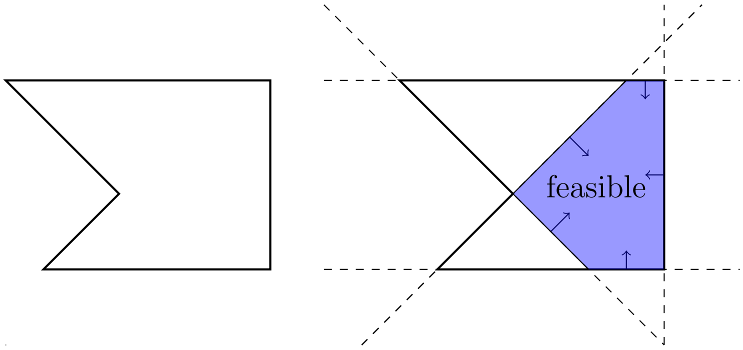
3 Non-convex stones

The major limitation of this formulation is that it cannot accurately model a rough stone. Nearly all rough diamond stones are non-convex, due to small dents or imperfections. There is little point in devising complicated solving methods to find a globally optimum diamond, if the underlying model of the stone is a poor approximation of reality. The method outlined in this section will provide the necessary foundation to model realistic rough diamond stones and non-convex diamonds once extended into three dimensions. Moreover, the methods established in this section are used to extend the diamond cutting model to include multiple diamonds.

The methods used to solve the convex diamond cutting problem in section 2 cannot directly be used for non-convex stones. Constraint set (18) requires every diamond vertex to lie inside the intersection of the set of half-spaces that define the rough stone. When applied to a non-convex stone, this constraint restricts the diamond from being placed in areas cut off by the extended edges of the stone that form the non-convex notch. This is shown in figure 5. The blue area where diamond vertices are allowed is considerably smaller than the area of the original non-convex stone.

3.1 Vertex Addition

Due to the issue demonstrated in figure 5, we must redefine the feasible region. Instead of requiring the diamond to lie within the intersection of all half-spaces (edges) of the stone, we instead require each point of the diamond to lie within only one half-space that forms a non-convex notch (but not all points have to lie in same half-space). However, there is an infinite number of points that form the diamond. We can relax this condition by only considering a discrete number of diamond vertices.



(a) Non-convex stone **(b)** Intersection of half-spaces formed from edges

Figure 5: Incorrect feasible region based on previous formulation.

3.1.1 Relaxed formulation

This is most clearly illustrated by an example. A solution to a relaxed problem of this type is shown in figure 6c, where we represent the diamond by a discrete number of vertices shown by circles. The edges labelled 1 and 2 form the non-convex notch. Note how diamond vertex A lies within the extension of edge 1, but not the extension of edge 2. Similarly vertex B meets the constraint associated with the extension of edge 2, but not edge 1. However, this solution is clearly not a final solution, because the diamond is not fully enclosed by the rough stone.

More formally, let us partition the half-spaces defining edges of the stone into different sets. Set N_p contains half-spaces that form a non-convex notch. For example, in figure 6b N_1 contains edges $1, 2$. Set C contains the edges the remaining edges $3, 4, 5, 6$. Finally set P contains the set of non-convex notches.

The problem is formally defined as follows

$$\max \quad \beta^2 + (\hat{s} - \beta)^2 \tag{24}$$

$$\text{s.t.} \quad a_j\tilde{x}_i + b_j\tilde{y}_i - MZ_k \leq c_j \qquad \forall i \in V, \forall j \in C, \forall k \in K \tag{25}$$

$$a_j\tilde{x}_i + b_j\tilde{y}_i - M(Z_k + Q_{i,j,p}) \leq c_j \quad \forall i \in V, \forall j \in N_p, \forall p \in P, \forall k \in K \tag{26}$$

$$\sum_{j \in N_p} Q_{i,j,p} = |N_p| - 1 \qquad \forall i \in V, \quad \forall p \in P \tag{27}$$

$$\sum_{k=1}^{4} Z_k = 3 \tag{28}$$

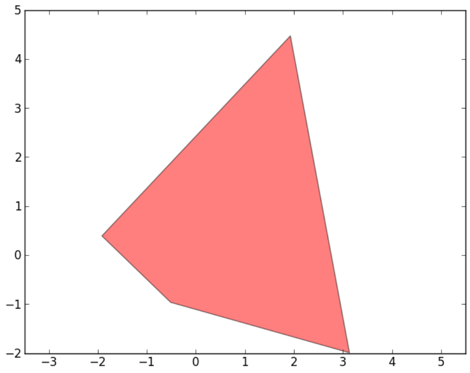
$$\tilde{x}_i = \beta x_i + (\hat{s} - \beta)\hat{x}_i + x_{off} \qquad \forall i \in V \tag{29}$$

$$\tilde{y}_i = \beta y_i + (\hat{s} - \beta)\hat{y}_i + y_{off} \qquad \forall i \in V \tag{30}$$

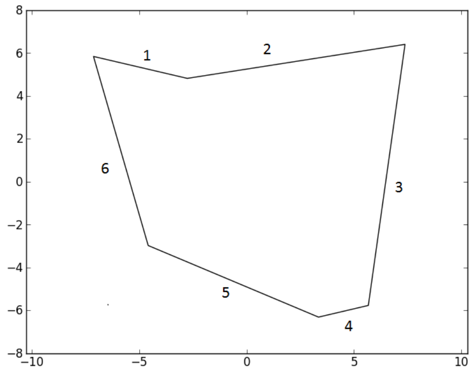
$$0 \leq \beta \leq \hat{s} \tag{31}$$

$$\hat{s} \geq 0 \qquad x_{off}, y_{off}, \tilde{x}_i, \tilde{y}_i \ \text{ free} \qquad Z_k \in 0,1 \quad Q_{i,j,p} \in 0,1 \tag{32}$$

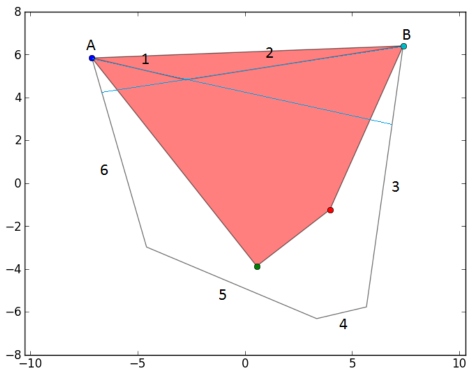
The objective (24) and constraints (29) and (30) that control the transformation of the diamond are the same as the weighted average formulation. Constraint (25) ensures all diamond vertices lie within all stone edges that are part of set C. Constraints (26), (27) and (28) require each diamond vertex to lie within one half-space that forms a non-convex notch. This is achieved using a set of binary variables $Q_{i,j,k}$.



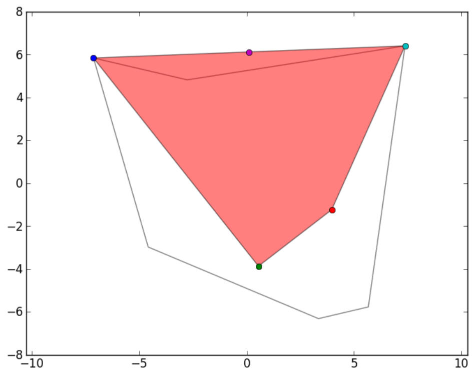
(a) Original diamond.



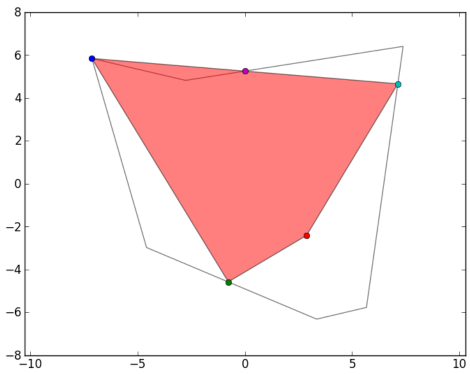
(b) Non-convex stone.



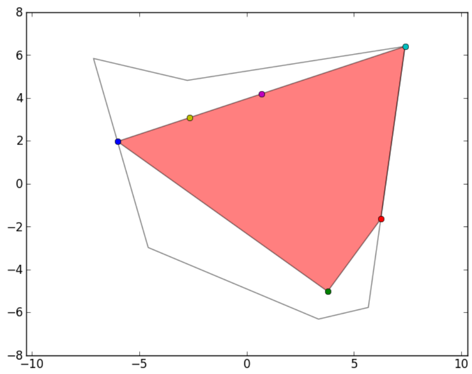
(c) Relaxed solution 1.



(d) Vertex to add.



(e) Next relaxed solution with vertex added



(f) Optimal solution.

Figure 6: Steps of the vertex addition method.

3.1.2 Detecting infeasibility

The solution to the relaxed formulation is not the final solution because part of the diamond lies outside the stone. This occurs when an edge of the diamond intersects an edge of the stone. In figure 6c, there are two points of intersection: at vertex B and near vertex A. To detect this, a line segment intersection algorithm was implemented. If there are no intersections, then the diamond is fully enclosed by the stone, and the algorithm can be terminated.

3.1.3 Regaining feasibility

To bring the diamond closer to feasibility another vertex is added to the diamond halfway between the points of intersection between the diamond and stone edges. The steps of this process are illustrated in figure 6. When the relaxed diamond cutting problem is solved again, the constraints force the diamond to shrink and re-orientate to fit the new vertex inside the rough stone. New vertices can be added iteratively, until the whole diamond lies within the stone. The final optimal solution is shown in figure 6f.

The vertex addition method solves a series of optimisation problems with linear constraints and a non-linear objective. Using the same process as section 2.3, given the approximate objective z, we have proved in Biggs (2013) that the global optimum objective value z^* will be within bounds given by $z - \frac{b^2}{2} \leq z^* \leq z$, where b relates to the interval width of the piecewise linear function.

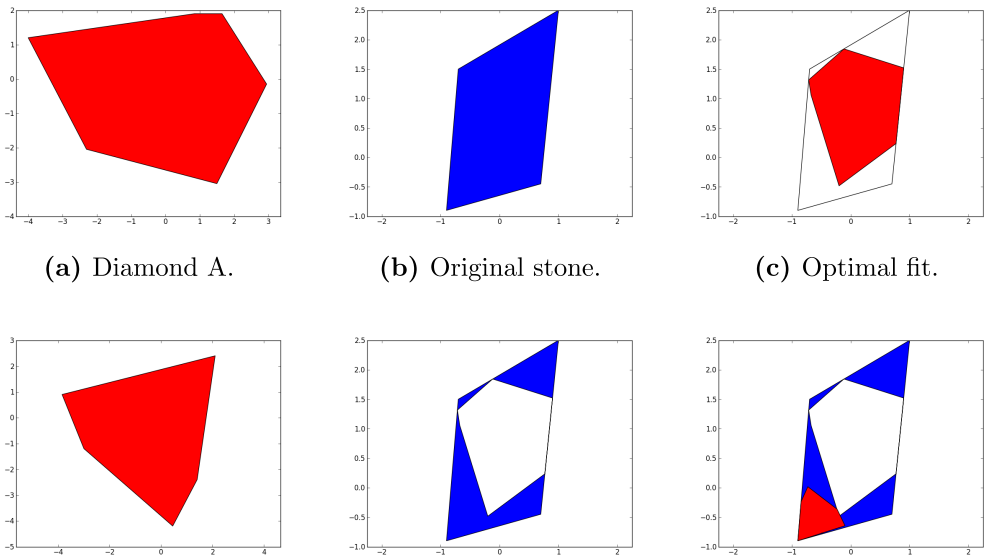
4 Multiple diamond cutting problem

While finding the largest diamond that fits inside a stone is useful, there are often more valuable ways to cut a diamond stone. This method of cutting gives no consideration to waste after this large diamond has been cut. The total volume may be utilised more effectively by cutting multiple diamonds, with a greater cumulative value than a single diamond.

We developed a greedy sequential algorithm to place multiple diamonds. This method builds on the vertex addition method. The algorithm finds the largest diamond that will fit in the convex stone, removes this diamond to leave a non-convex residual stone and places a second diamond in the remaining space. This process is illustrated in figure 7.

We then explored a more complex method which places two diamonds simultaneously. We formulated a relaxed multiple diamond formulation, that allows overlap between two diamonds, but requires the vertices of one diamond (known as the slave) to lie outside the other diamond(the master). By adding vertices to the slave diamond and resolving the problem, the relaxed solution can be strengthened, similar to the vertex addition method. We iterate towards a feasible solution where the two diamonds do not overlap. To linearise the constraints of this formulation, we fix the rotation of one diamond. The full formulation can be found in Biggs (2013).

This simultaneous method always produces diamonds with a greater combined area than the greedy method. In 24 percent of trials, the improvement was greater than a 1 percent increase in total area, up to 10 percent in some trials. Considering the significant value of diamonds, this occasional increase warrants further investigation. A full comparison of the models can be found in Biggs (2013).



(a) Diamond A. (b) Original stone. (c) Optimal fit.

(d) Diamond B. (e) Reduced stone. (f) Greedy solution.

Figure 7: Illustration of stages of greedy algorithm.

5 Conclusions

1. For a non-convex stone we devised an iterative vertex addition method that solves an integer problem with linear constraints and a non-linear objective.

2. We can find bounds on global optimum solution, dependent on the accuracy of a piecewise linear approximation to the objective.

3. The vertex addition method can be extended to place multiple diamonds using a greedy sequential heuristic or a simultaneous placement method.

4. The simultaneous method produces diamonds with a greater combined area and offers potential for development into three dimensions.

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