The Bi-objective Multi-Commodity Minimum Cost Flow Problem

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Abstract

While there are several algorithms for bi-objective single commodity flow problems there is only a single paper on the problem with two commodities. In this paper, by extending the idea of the mentioned paper, we present an approach which allows us to split the bi-objective three commodity minimum cost flow problem into four standard bi-objective minimum cost flow problems with a single commodity and solve them with a parametric network simplex algorithm. Based on results obtained for the two and three commodity problems we extend the algorithm to address the general multi-commodity case.

Key words: Multi-commodity flow, bi-objective minimum cost flow problem

1 Introduction

In most real-world optimisation problems, there is usually more than one objective as well as several different commodities that have to be taken into account. Thus, it would seem that multi-objective multi-commodity flow models would be more appropriate for modelling real-world decision making situations in the field of network optimization. While there are several algorithms for multi-objective single commodity network flow problems e.g. Raith and Ehrgott (2009) and Sedeño-Noda (2003), there is only a single paper on the problem with two commodities by Sedeño-Noda, Gonzalez-Martin, and Gutierrez (2005). Their main idea is introducing a change of variables in the formulation of the bi-objective two commodity minimum cost flow (B2CMCF) problem and to split the problem into two bi-objective minimum cost flow (BMCF) problems with a single commodity. We extend the idea and propose a change of variables in the bi-objective three commodity minimum cost flow (B3CMCF) problem which allows us to split the original problem into four standard BMCF problems. Based on results obtained for the two and three commodity problems, we extend the method to address the general multi-commodity case.

The paper is organised as follows: In Section 2, the bi-objective multi-commodity minimum cost flow (BMCMCF) problem is introduced. We present the change of variables approach for B3CMCF in Section 3. In Section 4, a method to solve BMCMCF is explained. Finally, in Section 5, we conclude the paper.

2 Bi-objective multi-commodity minimum cost flow problem

In this section, terminology and basic theory of BMCMCF problems are introduced. We will follow the same notation, definition and formulation as in Sedeño-Noda, Gonzalez-Martin, and Gutierrez (2005) and Hu (1963). Let $G = (V, A)$ be an antisymmetric directed network with a set of nodes or vertices $V = \{1, 2, \ldots, n\}$ and a set of arcs $A \subseteq V \times V$ with $|A| = m$. A directed network is antisymmetric if $(i, j) \in A \Rightarrow (j, i) \notin A$. There are q commodities sharing the capacity $u_{ij} \geqslant 0$ for each arc $(i, j) \in A$. We distinguish two special groups of nodes in G; the source nodes $S = \{s^1, s^2, \ldots, s^q\}$, and the sink nodes $T = \{t^1, t^2, \ldots, t^q\}$. For each commodity $k = 1, 2, \ldots, q$, b^k units of flow should be shipped from its source node s^k to its sink node t^k. Let $\left(c^k_{ij}, d^k_{ij}\right)$ be the pair of unit flow costs on arc (i, j) for commodity k and x^k_{ij} presents the amount of flow going through arc (i, j) for commodity k.

The BMCMCF problem is defined by the following mathematical program:

$$
\min \quad f(x) = \begin{cases} f^1(x) = \displaystyle\sum_{k=1,\ldots,q} \sum_{(i,j)\in A} c^k_{ij} x^k_{ij} \\ f^2(x) = \displaystyle\sum_{k=1,\ldots,q} \sum_{(i,j)\in A} d^k_{ij} x^k_{ij} \end{cases}
$$

$$
\text{s.t} \quad \sum_{\{j|(i,j)\in A\}} x^k_{ij} - \sum_{\{j|(j,i)\in A\}} x^k_{ij} = \begin{cases} b^k & \text{if } i \in S \\ 0 & \text{if } i \in V - \{S \cup T\}, \; k = 1, 2, \ldots, q \\ -b^k & \text{if } i \in T \end{cases}
$$

$$
\sum_{k=1,2,\ldots q} | x^k_{ij} | \leqslant u_{ij}, (i, j) \in A. \tag{2.1}
$$

For notational convenience, we will denote symmetric directed network $G' = (V', A')$ where $V' = V$ and A' contains the arcs (i, j) and (j, i), with capacities $u'_{ij} = u'_{ji} = u_{ij}$, for each arc $(i, j) \in A$. Figure 1 illustrates the two networks.

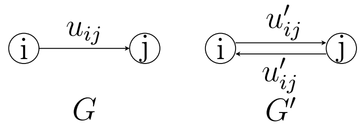

Figure 1: Network G' defined from the original network G

Definition 1. *Let X denote the feasible space for BMCMCF problem and $f(X) = \{(f^1(x), f^2(x)) \mid x \in X\}$ the objective space. A feasible solution $\hat{x} \in X$ of the BMCMCF problem is efficient if and only if there does not exist any $x' \in X$ with $f^i(x') \leqslant f^i(\hat{x})$ for both objectives and with $f^i(x') \neq f^i(\hat{x})$ for at least one i, and $i = 1, 2$.*

We will denote by $E[X]$ the set of efficient solutions of X and by $E[f(X)] = \{f(x) \mid x \in E[X]\}$ the set of non dominated points of $f(X)$. In this case, X is a compact polyhedron, and therefore $f(X)$ is also a compact polyhedron. Thus, $E[X]$ and $E[f(X)]$ through the set of efficient extreme points in the decision space $E_{ex}[X]$ and the objective space $E_{ex}[f(X)]$, respectively. Consequently, we are interested in obtaining $E_{ex}[f(X)]$ and $E_{ex}[X]$.

3 Change of variables method

Sedeño-Noda, Gonzalez-Martin, and Gutierrez (2005) and Sedeño-Noda, Gonzalez-Martin, and Alonso-Rodriguez (2008) presented a change of variables method in the formulation of the bi-objective two commodity minimum cost flow problem (B2CMCF), that splits the problem into two BMCF problems with a single commodity. We extend the idea and propose a change of variables in the bi-objective three commodity minimum cost flow (B3CMCF) problem which allows us to split the original problem into four standard BMCF problems.

3.1 Change of variables method for B2CMCF

In this subsection we briefly explain the method of Sedeño-Noda, Gonzalez-Martin, and Gutierrez (2005) for B2CMCF.
Let

$$x_{ij}^1 + x_{ij}^2 = z_{ij}^1 - z_{ji}^1,\ x_{ij}^1 - x_{ij}^2 = z_{ij}^2 - z_{ji}^2 \tag{3.1}$$

and hence,

$$x_{ij}^1 = \frac{z_{ij}^1 - z_{ji}^1 + z_{ij}^2 - z_{ji}^2}{2},\ x_{ij}^2 = \frac{z_{ij}^1 - z_{ji}^1 - z_{ij}^2 + z_{ji}^2}{2}. \tag{3.2}$$

Considering the above change of variables, the objective functions of the B2CMCF problem (2.1) for $q = 2$, can be formulated as follows:

$$f^1(x) = \sum_{(i,j)\in A} c_{ij}^1 x_{ij}^1 + c_{ij}^2 x_{ij}^2 = \frac{1}{2}\sum_{(i,j)\in A}\left(c_{ij}^1 + c_{ij}^2\right)\left(x_{ij}^1 + x_{ij}^2\right) + \frac{1}{2}\sum_{(i,j)\in A}\left(c_{ij}^1 - c_{ij}^2\right)\left(x_{ij}^1 - x_{ij}^2\right),$$

$$f^2(x) = \sum_{(i,j)\in A} d_{ij}^1 x_{ij}^1 + d_{ij}^2 x_{ij}^2 = \frac{1}{2}\sum_{(i,j)\in A}\left(d_{ij}^1 + d_{ij}^2\right)\left(x_{ij}^1 + x_{ij}^2\right) + \frac{1}{2}\sum_{(i,j)\in A}\left(d_{ij}^1 - d_{ij}^2\right)\left(x_{ij}^1 - x_{ij}^2\right).$$

To simplify the formulation, α_{ij}^p and β_{ij}^p are introduced as costs associated with the arc $(i,j) \in A'$ and the pth objective $(p = 1, 2)$. They are given by

$$\alpha_{ij}^1 = \begin{cases} \frac{1}{2}\left(c_{ij}^1 + c_{ij}^2\right) & \text{if } (i,j) \in \text{A}, \\ -\frac{1}{2}\left(c_{ji}^1 + c_{ji}^2\right) & \text{if } (j,i) \in \text{A}, \end{cases} \quad \text{and} \quad \alpha_{ij}^2 = \begin{cases} \frac{1}{2}\left(d_{ij}^1 + d_{ij}^2\right) & \text{if } (i,j) \in \text{A}, \\ -\frac{1}{2}\left(d_{ji}^1 + d_{ji}^2\right) & \text{if } (j,i) \in \text{A}, \end{cases} \quad \text{and}$$

$$\beta_{ij}^1 = \begin{cases} \frac{1}{2}\left(c_{ij}^1 - c_{ij}^2\right) & \text{if } (i,j) \in \text{A}, \\ -\frac{1}{2}\left(c_{ji}^1 - c_{ji}^2\right) & \text{if } (j,i) \in \text{A}, \end{cases} \quad \text{and} \quad \beta_{ij}^2 = \begin{cases} \frac{1}{2}\left(d_{ij}^1 - d_{ij}^2\right) & \text{if } (i,j) \in \text{A}, \\ -\frac{1}{2}\left(d_{ji}^1 - d_{ji}^2\right) & \text{if } (j,i) \in \text{A}. \end{cases}$$

Thus the objective functions become as follows:

$$f^1(x) = \sum_{(i,j)\in A'} \alpha_{ij}^1 z_{ij}^1 + \sum_{(i,j)\in A'} \beta_{ij}^1 z_{ij}^2 = f^1\left(z^1\right) + f^1\left(z^2\right),$$

$$f^2(x) = \sum_{(i,j)\in A'} \alpha_{ij}^2 z_{ij}^1 + \sum_{(i,j)\in A'} \beta_{ij}^2 z_{ij}^2 = f^2\left(z^1\right) + f^2\left(z^2\right).$$

For each fixed $i \in V$, the flow conservation constraints (2.1) are added and subtracted for $k = 1, 2$. Then making the change of variables, and considering that in the antisymmetric network G we have $\{j \mid (i,j) \in A\} \cap \{j \mid (j,i) \in A\} = \emptyset$ we obtain:

$$\sum_{\{j|(i,j)\in A'\}} z_{ij}^p - \sum_{\{j|(j,i)\in A'\}} z_{ji}^p = \begin{cases} b^1 & \text{if } i = s^1 \\ (-1)^{p-1} b^2 & \text{if } i = s^2 \\ 0 & \text{if } i \in V - \{s^1, s^2, t^1, t^2\} \\ -b^1 & \text{if } i = t^1 \\ (-1)^{p-1} b^2 & \text{if } i = t^2. \end{cases}, \; p = 1, 2.$$

Also for constraints (2.1) we can write $0 \leqslant z_{ij}^p \leqslant u'_{ij}$ for all arcs $(i,j) \in A'$ and $p = 1, 2$.

Applying the change of variables approach, the B2CMCF problem can be reformulated to give

$$\begin{aligned} \min f(x) &= \left(\sum_{(i,j)\in A'} \alpha_{ij}^1 z_{ij}^1 + \sum_{(i,j)\in A'} \beta_{ij}^1 z_{ij}^2, \sum_{(i,j)\in A'} \alpha_{ij}^2 z_{ij}^1 + \sum_{(i,j)\in A'} \beta_{ij}^2 z_{ij}^2 \right) = \\ & \left(f^1(z^1) + f^1(z^2), f^2(z^1) + f^2(z^2) \right) \quad (3.3) \end{aligned}$$

$$\text{s.t:} \quad \sum_{\{j|(i,j)\in A'\}} z_{ij}^1 - \sum_{\{j|(j,i)\in A'\}} z_{ji}^1 = \begin{cases} b^1 & \text{if } i = s^1 \\ b^2 & \text{if } i = s^2 \\ 0 & \text{if } i \in V - \{s^1, s^2, t^1, t^2\}, \\ -b^1 & \text{if } i = t^1 \\ -b^2 & \text{if } i = t^2 \end{cases} \quad (3.4)$$

$$\sum_{\{j|(i,j)\in A'\}} z_{ij}^2 - \sum_{\{j|(j,i)\in A'\}} z_{ji}^2 = \begin{cases} b^1 & \text{if } i = s^1 \\ -b^2 & \text{if } i = s^2 \\ 0 & \text{if } i \in V - \{s^1, s^2, t^1, t^2\}, \\ -b^1 & \text{if } i = t^1 \\ b^2 & \text{if } i = t^2 \end{cases} \quad (3.5)$$

$$0 \leqslant z_{ij}^p \leqslant u'_{ij}, p = 1, 2 \forall (i,j) \in A'. \quad (3.6)$$

Now the objective functions and the constraints of the above formulation can be separated in accordance with the variables z^p. According to the Proposition 1 in Sedeño-Noda, Gonzalez-Martin, and Gutierrez (2005), a solution of the above formulation is derived from the Cartesian product of the efficient sets of two classical BMCF problems, the first one with objective functions $\min \left(f^1(z^1), f^2(z^1)\right)$ with constraints (3.4) and (3.6) and the second one with objective functions $\min \left(f^1(z^2), f^2(z^2)\right)$ and constraints (3.5) and (3.6).

3.2 Change of variables method for B3MCF

Using the BMCMCF formulation (2.1) with $q = 3$, the B3CMCF problem can be stated as follows

$$\min z = \begin{cases} f^1(x) = & \sum_{(i,j)\in A} c^1_{ij}x^1_{ij} + c^2_{ij}x^2_{ij} + c^3_{ij}x^3_{ij} \\ f^2(x) = & \sum_{(i,j)\in A} d^1_{ij}x^1_{ij} + d^2_{ij}x^2_{ij} + d^3_{ij}x^3_{ij} \end{cases} \tag{3.7}$$

$$\text{s.t.} \quad \sum_{\{j|(i,j)\in A\}} x^k_{ij} - \sum_{\{j|(j,i)\in A\}} x^k_{ij} = \begin{cases} b^k & \text{if } i = s^k \\ 0 & \text{if } i \in V - \{s^k, t^k\}, k = 1,2,3 \\ -b^k & \text{if } i = t^k \end{cases} \tag{3.8}$$

$$\sum_{k=1,2,3} | x^k_{ij} | \leqslant u_{ij}, (i,j) \in A. \tag{3.9}$$

Now we propose the following change of variables which allows us to split the above problem to four standard BMCF problems with a single commodity. Let

$$\begin{aligned} x^1_{ij} + x^2_{ij} + x^3_{ij} = z^1_{ij} - z^1_{ji} \\ x^1_{ij} + x^2_{ij} - x^3_{ij} = z^2_{ij} - z^2_{ji} \\ x^1_{ij} - x^2_{ij} + x^3_{ij} = z^3_{ij} - z^3_{ji} \\ x^1_{ij} - x^2_{ij} - x^3_{ij} = z^4_{ij} - z^4_{ji} \end{aligned} \tag{3.10}$$

and hence:

$$\begin{aligned} x^1_{ij} &= \frac{z^1_{ij} - z^1_{ji} + z^2_{ij} - z^2_{ji} + z^3_{ij} - z^3_{ji} + z^4_{ij} - z^4_{ji}}{4} \\ x^2_{ij} &= \frac{z^1_{ij} - z^1_{ji} + z^2_{ij} - z^2_{ji} - z^3_{ij} + z^3_{ji} - z^4_{ij} + z^4_{ji}}{4} \\ x^3_{ij} &= \frac{z^1_{ij} - z^1_{ji} - z^2_{ij} + z^2_{ji} + z^3_{ij} - z^3_{ji} - z^4_{ij} + z^4_{ji}}{4}. \end{aligned} \tag{3.11}$$

Considering the above change of variables, the objective function $f^1(x)$ of the B3CMCF problem in equation (3.7) can be formulated as follows:

$$\begin{aligned} & f^1(x) = \sum_{(i,j)\in A} c^1_{ij}x^1_{ij} + c^2_{ij}x^2_{ij} + c^3_{ij}x^3_{ij} = \\ & \frac{1}{4}\sum_{(i,j)\in A} \left(c^1_{ij} + c^2_{ij} + c^3_{ij}\right)\left(x^1_{ij} + x^2_{ij} + x^3_{ij}\right) + \\ & \frac{1}{4}\sum_{(i,j)\in A} \left(c^1_{ij} + c^2_{ij} - c^3_{ij}\right)\left(x^1_{ij} + x^2_{ij} - x^3_{ij}\right) + \\ & \frac{1}{4}\sum_{(i,j)\in A} \left(c^1_{ij} - c^2_{ij} + c^3_{ij}\right)\left(x^1_{ij} - x^2_{ij} + x^3_{ij}\right) + \\ & \frac{1}{4}\sum_{(i,j)\in A} \left(c^1_{ij} - c^2_{ij} - c^3_{ij}\right)\left(x^1_{ij} - x^2_{ij} - x^3_{ij}\right). \end{aligned} \tag{3.12}$$

To simplify the formulation (3.12) consider α^1_{ij}, β^2_{ij}, γ^1_{ij} and δ^1_{ij} as the costs associated with the arc $(i,j) \in A'$ and the objective $f^1(x)$, and given by :

$$\alpha_{ij}^1 = \begin{cases} \frac{1}{4}\left(c_{ij}^1 + c_{ij}^2 + c_{ij}^3\right) & \text{if } (i,j) \in \text{A}, \\ -\frac{1}{4}\left(c_{ji}^1 + c_{ji}^2 + c_{ji}^3\right) & \text{if } (j,i) \in \text{A}, \end{cases} \quad \text{and} \quad \beta_{ij}^1 = \begin{cases} \frac{1}{4}\left(c_{ij}^1 + c_{ij}^2 + -c_{ij}^3\right) & \text{if } (i,j) \in \text{A}, \\ -\frac{1}{4}\left(c_{ji}^1 + c_{ji}^2 - c_{ji}^3\right) & \text{if } (j,i) \in \text{A}, \end{cases}$$

$$\gamma_{ij}^1 = \begin{cases} \frac{1}{4}\left(c_{ij}^1 - c_{ij}^2 + c_{ij}^3\right) & \text{if } (i,j) \in \text{A}, \\ -\frac{1}{4}\left(c_{ji}^1 - c_{ji}^2 + c_{ji}^3\right) & \text{if } (j,i) \in \text{A}, \end{cases} \quad \text{and} \quad \delta_{ij}^1 = \begin{cases} \frac{1}{4}\left(c_{ij}^1 - c_{ij}^2 - c_{ij}^3\right) & \text{if } (i,j) \in \text{A}, \\ -\frac{1}{4}\left(c_{ji}^1 - c_{ji}^2 - c_{ji}^3\right) & \text{if } (j,i) \in \text{A}. \end{cases}$$

Thus the objective function $f^1(x)$ becomes

$$f^1(x) = \sum_{(i,j)\in A'} \alpha_{ij}^1 z_{ij}^1 + \sum_{(i,j)\in A'} \beta_{ij}^1 z_{ij}^2 + \sum_{(i,j)\in A'} \gamma_{ij}^1 z_{ij}^3 + \sum_{(i,j)\in A'} \delta_{ij}^1 z_{ij}^4 = f^1\left(z^1\right) + f^1\left(z^2\right) + f^1\left(z^3\right) + f^1\left(z^4\right). \tag{3.13}$$

Doing the same things for the objective function $f^2(x)$ we can rewrite the function as follows:

$$f^2(x) = \sum_{(i,j)\in A'} \alpha_{ij}^2 z_{ij}^1 + \sum_{(i,j)\in A'} \beta_{ij}^2 z_{ij}^2 + \sum_{(i,j)\in A'} \gamma_{ij}^2 z_{ij}^3 + \sum_{(i,j)\in A'} \delta_{ij}^2 z_{ij}^4 = f^2\left(z^1\right) + f^2\left(z^2\right) + f^2\left(z^3\right) + f^2\left(z^4\right). \tag{3.14}$$

For each fixed $i \in V$, the flow conservation constraints (3.8) are combined by using different combinations of addition and subtraction, similar to the multipliers of x_{ij}^k from (3.11), for $k = 1, 2, 3$. Then making the change of variables, we obtain the following constraints:

$$\sum_{\{j|(i,j)\in A'\}} z_{ij}^1 - \sum_{\{j|(j,i)\in A'\}} z_{ji}^1 = \begin{cases} b^1 & \text{if } i = s^1 \\ b^2 & \text{if } i = s^2 \\ b^3 & \text{if } i = s^3 \\ 0 & \text{if } i \in V - \{s^1, s^2, s^3, t^1, t^2, t^3\} \\ -b^1 & \text{if } i = t^1 \\ -b^2 & \text{if } i = t^2 \\ -b^3 & \text{if } i = t^3 \end{cases} \tag{3.15}$$

$$\sum_{\{j|(i,j)\in A'\}} z_{ij}^2 - \sum_{\{j|(j,i)\in A'\}} z_{ji}^2 = \begin{cases} b^1 & \text{if } i = s^1 \\ b^2 & \text{if } i = s^2 \\ -b^3 & \text{if } i = s^3 \\ 0 & \text{if } i \in V - \{s^1, s^2, s^3, t^1, t^2, t^3\} \\ -b^1 & \text{if } i = t^1 \\ -b^2 & \text{if } i = t^2 \\ b^3 & \text{if } i = t^3 \end{cases} \tag{3.16}$$

$$\sum_{\{j|(i,j)\in A'\}} z_{ij}^3 - \sum_{\{j|(j,i)\in A'\}} z_{ji}^3 = \begin{cases} b^1 & \text{if } i = s^1 \\ -b^2 & \text{if } i = s^2 \\ b^3 & \text{if } i = s^3 \\ 0 & \text{if } i \in V - \{s^1, s^2, s^3, t^1, t^2, t^3\} \\ -b^1 & \text{if } i = t^1 \\ b^2 & \text{if } i = t^2 \\ -b^3 & \text{if } i = t^3 \end{cases} \tag{3.17}$$

$$\sum_{\{j|(i,j)\in A'\}} z_{ij}^4 - \sum_{\{j|(j,i)\in A'\}} z_{ji}^4 = \begin{cases} b^1 & \text{if } i = s^1 \\ -b^2 & \text{if } i = s^2 \\ -b^3 & \text{if } i = s^3 \\ 0 & \text{if } i \in V - \{s^1, s^2, s^3, t^1, t^2, t^3\} \\ -b^1 & \text{if } i = t^1 \\ b^2 & \text{if } i = t^2 \\ b^3 & \text{if } i = t^3. \end{cases} \tag{3.18}$$

For constraint (3.9) we have:

$$\sum_{k=1,2,3} | x_{ij}^k | \leqslant u_{ij}, \iff \begin{cases} -u_{ij} \leqslant x_{ij}^1 + x_{ij}^2 + x_{ij}^3 \leqslant u_{ij} \\ -u_{ij} \leqslant x_{ij}^1 + x_{ij}^2 - x_{ij}^3 \leqslant u_{ij} \\ -u_{ij} \leqslant x_{ij}^1 - x_{ij}^2 + x_{ij}^3 \leqslant u_{ij} \\ -u_{ij} \leqslant x_{ij}^1 - x_{ij}^2 - x_{ij}^3 \leqslant u_{ij}. \end{cases}$$

After changing the variables the new set of constraints follows $-u_{ij} \leqslant z_{ij}^p - z_{ji}^p \leqslant u_{ij}$ for $p = 1, 2, 3, 4$. And for symmetric network G', the bounds are stated as $0 \leqslant z_{ij}^p \leqslant u'_{ij}$ for all arcs $(i, j) \in A'$ and $p = 1, 2, 3, 4$.

Applying the above change of variables approach the B3CMCF problem can be split into the four following problems:

$$\begin{aligned} \min \quad & \left(f^1\left(z^1\right), f^2\left(z^1\right)\right) = \left(\sum_{(i,j)\in A'} \alpha_{ij}^1 z_{ij}^1, \sum_{(i,j)\in A'} \alpha_{ij}^2 z_{ij}^1\right) \\ \text{s.t:} \quad & \text{constraints (3.15)} \\ & 0 \leqslant z_{ij}^1 \leqslant u'_{ij}, \forall (i,j) \in A' \end{aligned} \tag{3.19}$$

$$\begin{aligned} \min \quad & \left(f^1\left(z^2\right), f^2\left(z^2\right)\right) = \left(\sum_{(i,j)\in A'} \beta_{ij}^1 z_{ij}^2, \sum_{(i,j)\in A'} \beta_{ij}^2 z_{ij}^2\right) \\ \text{s.t:} \quad & \text{constraints (3.16)} \\ & 0 \leqslant z_{ij}^2 \leqslant u'_{ij}, \forall (i,j) \in A' \end{aligned} \tag{3.20}$$

$$\begin{aligned}
\min \quad & \left(f^1\left(z^3\right), f^2\left(z^3\right)\right) = \left(\sum_{(i,j)\in A'} \gamma_{ij}^1 z_{ij}^3, \sum_{(i,j)\in A'} \gamma_{ij}^2 z_{ij}^3\right) \\
\text{s.t:} \quad & \text{constraints (3.17)} \\
& 0 \leqslant z_{ij}^3 \leqslant u'_{ij}, \forall (i,j) \in A'
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
\min \quad & \left(f^1\left(z^4\right), f^2\left(z^4\right)\right) = \left(\sum_{(i,j)\in A'} \delta_{ij}^1 z_{ij}^4, \sum_{(i,j)\in A'} \delta_{ij}^2 z_{ij}^4\right) \\
\text{s.t:} \quad & \text{constraints (3.18)} \\
& 0 \leqslant z_{ij}^4 \leqslant u'_{ij}, \forall (i,j) \in A'
\end{aligned} \tag{3.22}$$

The four above problems are classical BMCF problems with one commodity which can be solved by several existing algorithms. Then undoing the change of variables the efficient set of the B3CMCF problem can be derived.

3.3 Parametric simplex algorithm

As mentioned above, the change of variables method splits the B3CMCF problem into four BMCF problems with one single commodity which can be solved by any bi-objective one commodity minimum cost flow solution method. We used the parametric simplex method, as in the phase 1 algorithm used by Raith and Ehrgott (2009). In this method, one of the two lexicographically optimal solutions, e.g., the $lex\,(1,2)$-best solution, is obtained with a single-objective network simplex algorithm. As there are two cost components associated with each arc in the network simplex algorithm, the reduced cost of each arc also consists of two components. In each iteration of the network simplex algorithm, candidate entering arcs are selected with minimal ratio of their reduced cost derived from the current supported efficient solution. By doing so, the procedure generates a complete set of extreme efficient solutions moving in a left-to-right fashion. The parametric simplex algorithm finishes when no candidate arcs to enter the basis can be found which indicates that the $lex\,(2,1)$ best solution is obtained.

3.4 An example

Let us consider the example depicted in Figure 2. By applying the change of variables method for this B3CMCF problem, four standard BMCF problems with one commodity are obtained as shown in Figure 3, then we solved these four problems with the network simplex algorithm.

In Table 1, the efficient extreme points in the objective space for these four problems with their corresponding values $\left(f^1\left(z^p\right), f^2\left(z^p\right)\right)$, $p = 1,2,3,4$ are shown.

In Table 2, the three commodity efficient extreme points in the objective spaces with their corresponding values $\left(f^1\left(x\right), f^2\left(x\right)\right)$ are shown.

Table 1: Efficient extreme points for the one commodity problems

	$(s^1,1)$	$(1,s^1)$	$(s^2,1)$	$(1,s^2)$	$(s^3,1)$	$(1,s^3)$	$(1,2)$	$(2,1)$	$(2,t^1)$	$(t^1,2)$	$(2,t^2)$	$(t^2,2)$	$(2,t^3)$	$(t^3,2)$	(s^1,t^1)	(t^1,s^1)	(s^3,t^3)	(t^3,s^3)	$f^1(z)$	$f^2(z)$	$\lambda^{t-1} \leqslant \lambda \leqslant \lambda^t$
Efficient extreme points for the (BP1) problem																					
$(z^1)^1$	0	12	20	0	0	5	10	7	0	5	20	10	0	12	30	0	30	0	159.25	305.5	$0 \leqslant \lambda \leqslant 1$
Efficient extreme points for the (BP2) problem																					
$(z^2)^1$	0	12	20	0	2	20	0	10	0	18	20	0	8	20	30	0	0	7	96.5	-0.5	$0 \leqslant \lambda \leqslant 1$
Efficient extreme points for the (BP3) problem																					
$(z^3)^1$	0	10	0	20	20	0	0	10	20	0	0	20	10	20	28	0	30	25	25.5	29.5	$0.56 \leqslant \lambda \leqslant 1$
$(z^3)^2$	15	0	0	20	0	5	0	10	0	5	0	20	20	5	3	0	30	0	50.5	-1.75	$0 \leqslant \lambda \leqslant 0.56$
Efficient extreme points for the (BP4) problem																					
$(z^4)^1$	20	0	0	20	10	20	0	10	0	10	0	20	20	0	0	2	0	15	-4	34.5	$0.67 \leqslant \lambda \leqslant 1$
$(z^4)^2$	20	0	0	20	20	15	10	5	5	0	0	20	20	0	0	2	0	30	0.25	27	$0 \leqslant \lambda \leqslant 0.67$

Table 2: Efficient extreme points for the main three commodity problem

	$(s^1,1)$	$(s^2,1)$	$(s^3,1)$	$(1,2)$	$(2,t^1)$	$(2,t^2)$	$(2,t^3)$	(s^1,t^1)	(s^3,t^3)	$f^1(x)$	$f^2(x)$	$\lambda^{t-1} \leqslant \lambda \leqslant \lambda^t$
x_1	(-3.5,-8.5,-7.5)	(0,20,0)	(-3.25,-8.25,7)	(-6.75,3.25,-0.5)	(-3.25,-8.25,7)	(0,20,0)	(-3.5,-8.5,-7.5)	(21.5,8.5,7.5)	(3.25,8.25,18)	113	330.25	$0.67 \leqslant \lambda \leqslant 1$
x_2	(-3.5,-8.5,-7.5)	(0,20,0)	(0.5,-12,7)	(-3,-0.5,-0.5)	(0.5,-12,7)	(0,20,0)	(-3.5,-8.5,-7.5)	(21.5,8.5,7.5)	(-0.5,12,18)	88	361.5	$0.56 \leqslant \lambda \leqslant 0.67$
x_3	(2.75,-14.75,-1.25)	(0,20,0)	(-5.75,-5.75,0.25)	(-3,-0.5,-0.5)	(-5.75,-5.75,0.75)	(0,20,0)	(2.75,-14.75,-1.25)	(15.25,14.75,1.25)	(5.75,5.75,2425)	113	330.25	$0 \leqslant \lambda \leqslant 0.56$

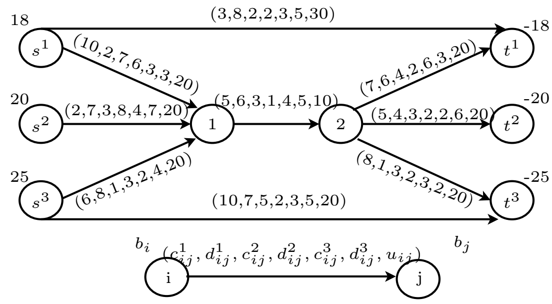

Figure 2: B3CMCF example

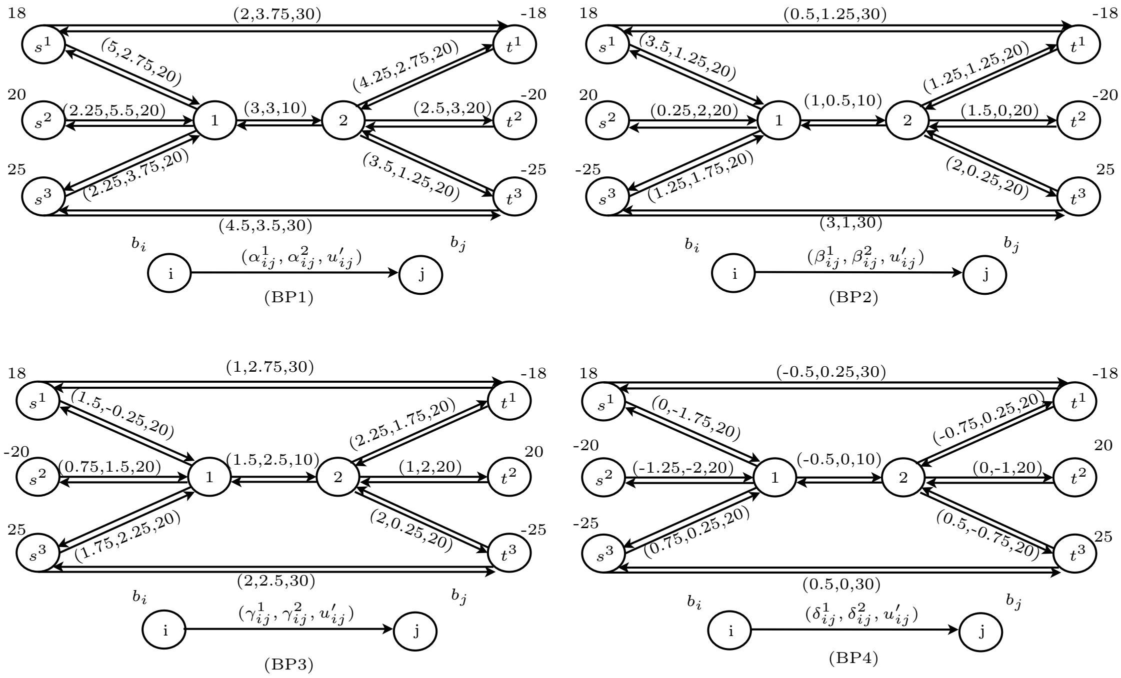

Figure 3: Networks with one commodity obtained from the B3CMCF problem

4 Change of variables method for BMCMCF

Based on results obtained for two and three commodity problems we can extend the change of variables method to address the general multi-commodity case. In this extension to solve the BMCMCF problem with q commodities ($q \geqslant 3$), by combining $x^k_{ij}, k = 1, 2, ..., q$ using different combinations of addition and subtraction in formulation (2.1) and introducing 2^{q-1} new variables $z^p_{ij}, p = 1, 2, \ldots, 2^{q-1}$, similar to (3.10), the BMCMCF problem can be split into 2^{q-1} classical BMCF problems with single commodity. These 2^{q-1} problems can be solved by different standard algorithms and then by undoing the change of variables the efficient set of the BMCMCF problem can be derived.

5 Conclusion

In this paper we have presented a method to solve the bi-objective multi-commodity minimum cost flow problem. For that we introduced a change of variables method which splits the BMCMCF problem with q commodities into 2^{q-1} classical BMCF problems with a single commodity, that can be solved by different standard algorithms.

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