Collateralised Debt Obligation Portfolio Optimisation

Min Gih Choi
Department of Engineering Science
University of Auckland
New Zealand
mcho099@aucklanduni.ac.nz

Abstract

Billions of dollars worth of CDO portfolios are issued every year, with individual portfolios being worth hundreds of millions of dollars. Such large investments need to be managed, and conventionally this has been done manually under the discretion of a manager proposing potential trades. Managing these portfolios requires balancing various metrics designed to measure the performance of the portfolio. A complex calculation which would provide an indication of the value of the portfolio can be carried out, known as the Par Coverage value. Under association with leading asset manager, M&G Investments, such calculations were reformulated into a mixed integer linear programme (MILP), in order to identify a more efficient portfolio, through maximising the Par Coverage value of any given CDO portfolio. The implementation of the optimisation showed promising results and was able to identify an efficient frontier of potential portfolios.

Key words: Collateralised Debt Obligations, CDO, Portfolio Optimisation, Par Coverage value, Weighted Average Spread, Weighted Average Rating Factor, Efficient Frontier.

1 Introduction

Collateralised Debt Obligations (CDOs) are a type of investment that is common in the finance industry. A CDO portfolio can be constructed through combining multiple individual *obligations* (otherwise known as assets or investments). Most CDO portfolios are actively managed to ensure the overall health of the portfolio is maintained despite fluctuations in the quality of the individual obligations. The management of a CDO is done through buying and selling obligations. In order to prevent bad choices and to keep the quality of the CDO within the conditions offered to the investors, there exist certain criteria and measures of portfolio performance (Fabozzi 2002). Common criteria and measures include the *Par Coverage Value* test, the *Weighted Average Spread* (WAS) trigger level, and the *Weighted Average Rating Factor* (WARF) trigger level.

1.1 Aim

Given a list of obligations that could be added to a portfolio, and an initial starting portfolio, the aim was to identify more efficient portfolios by buying and/or selling individual obligations. The problem was converted into a mixed integer linear problem (MILP) and solved with a MILP solver. As the manner in which a CDO portfolio is operated and managed varies depending on the type of CDOs and style of the portfolio manager, this paper only deals with a particular type of CDO under advice from a sponsoring company, M&G Investments. In modelling this CDO portfolio problem, the following four calculations are taken into account. the Par Coverage value, Weighted Average Spread (WAS), Weighted Average Rating Factor (WARF), and the principal cash sum. The Par Coverage value was selected as the first indicator to optimise over due to its complexity. The formulation of the MILP can be summarised as:

Maximise:	Par Coverage Value		
Subject to:	WAS	$\geq$	*minimum WAS level*
	WARF	$\leq$	*maximum WARF level*
	Principal Cash	$\geq$	*minimum cash*

2 Modelling a CDO

2.1 Parameters

The input would be provided in the form of a list of obligations. This list includes all obligations that are currently in the portfolio, and further appended are obligations that are available to purchase into the portfolio. Each obligation in the list will have associated with it an array of attributes identifying things ranging from the value of the obligation, to whether or not the obligation has defaulted (see Table 1). These attributes are necessary for the calculation of the values in the MILP and are inputted as the following parameters:

Parameter	Definition	Type
isD	*Defaulted Obligation:* an obligation that has failed to pay back a loan.	Binary
isLD	*Long Dated Obligation:* the maturity date of the obligation is past a predefined date.	Binary
isZC	*Zero Coupon Securities:* obligations that do not make periodic interest payments, and instead pay all interest at the end of maturity.	Binary
isDisc	*Discount Obligation:* obligation was purchased at less than a predefined threshold of the principal amount.	Binary
isN	*Normal Obligation:* when an obligation is not a Defaulted, Long Dated, or Zero Coupon obligation.	Binary
isCCC	*CCC Obligation:* has a credit rating that is considered CCC or below. They are considered highly risky investments.	Binary
inPort	Obligation was in the portfolio before optimisation.	Binary
MR	*Moody's Recovery:* the recovery rate as advised by Moody's Investor Service. The value of Long Dated obligations.	Continuous
SPR	*S&P recovery:* the recovery rate as advised by Standard & Poor’s.	Continuous
AV	*Accreted Value:* the value of Zero Coupon securities.	Continuous
PP	*Purchase Price:* the price of Discount Obligations.	Continuous

MV	*Market Value:* the determined bid price.	Continuous
WV	*Worst Value:* Minimum of the *MR*, *SPE*, and *MV*. The value of Defaulted Obligations	Continuous
CCCMV	*CCC market value:* re-calculated *MV* for CCC obligations.	Continuous
RF	*Moody's Rating Factor:* the credit rating factor as determined by Moody's Investor Service.	Integer
SPE	*S&P rating Factor Equivalent:* the credit rating factor as determined by Standard & Poor's.	Integer
Spread	*Spread:* The rate the loan is paying above the inter-bank lending rate (percentage).	Continuous
Buy	*Cost of Buying* the obligation.	Continuous
Sell	*Cost of Selling* the obligation.	Continuous
B	*Block Size:* obligations will usually trade in defined blocks, consisting of a certain monetary value. Purchases must be bought in blocks (e.g. purchasing six $10,000 blocks, equals $60,000).	Continuous
FV	*Total Nominal/Principal/Face Value* of the obligation.	Continuous
M	*Maximum blocks:* limited by the Total Face Value. $M = \lfloor FV/B \rfloor$.	Integer

Table 1. Parameters per obligation for the CDO portfolio problem.

Additionally, there are additional parameters pertaining to the entire portfolio as follows:

Parameter	**Definition**	**Type**
ipc	*Initial principal cash*	Continuous
minimum WAS	*WAS minimum,* trigger level for the portfolio	Continuous
maximum WARF	*WARF maximum,* trigger level for the WARF	Continuous
minimum cash	Minimum amount of principal cash the portfolio can have.	Continuous

Table 2. Additional parameters for the CDO portfolio problem.

2.2 Processes Before Solving

Before solving for the MILP, it is necessary to manipulate the input parameters in a form that is applicable for the linear formulation. All the binary parameters; *isD*, *isLD*, *isZC*, *isN*, *isDisc*, *isCCC*, and *inPort* indicate categories that the obligations can be in, and each category has associated with it a multiplier; *WV*, *MR*, *AV*, 1, *CCCMV*, and *PP* respectively, and no multiplier for the special category of *inPort*. It is necessary for the first 4 categories to be mutually exclusive. This is done by comparing the respective multipliers of the categories the obligation is in, and finding the lowest value. The obligation will be considered being in this category only, and no longer in any of the other three. All obligations must be in exactly one of these first 4 categories.

The categories *isDisc*, and *isCCC*, also need to be mutually exclusive and similarly, the respective multipliers are compared and the lowest selected. It is possible for an obligation to be neither of these 2 categories, or one of these categories, but not both.

The special category of *inPort* indicates which obligations were originally in the portfolio, and no changes are made to this category.

Finally all obligations in the entire list of obligations need to be ordered by *CCCMV*, such that the i^{th} obligation would always have a *CCCMV* equal or smaller to the j^{th} obligation for $j = i, i+1, ..., n$.

2.3 Problem Definition

The Par Coverage value, WAS, WARF, and Principal Cash sum calculations are as follows. Certain calculations can differ depending on the type of CDO portfolio, and the style of the management. Therefore the calculations utilised were as advised from M&G Investments for a specific Par Coverage value calculation.

For calculating the Par Coverage value of a given portfolio, first the value of all obligations in each of the categories of *isD*, *isLD*, *isZC*, and *isN* are summed up multiplied with the their respective multipliers to form a value known as the Aggregate Collateral Balance (ACB). The value of all obligations that are in the category of CCC obligations are also summed together to form a value known as the Aggregate Principal Balance of CCC obligations (APBCCC). If the APBCCC is larger than 7.5% × ACB, then the amount that is in excess is known as the CCC excess size. If it is less, then the following calculations for the CCC obligations are not performed.

A special subset of CCC obligations are selected (known as the CCC excess list) such that the aggregate value of this subset is equal to, this CCC excess size. The subset must be chosen in ascending order of *CCCMV* value. It is unlikely that the chosen subset will fit exactly within the CCC excess size, and so the last obligation in the subset will specifically be covering the threshold defined by the 7.5% × ACB. For this special obligation, only the fraction of that is included in contributing to the CCC excess size is considered (see Figure 1). If the list fits exactly, then the fractional proportion is 0.

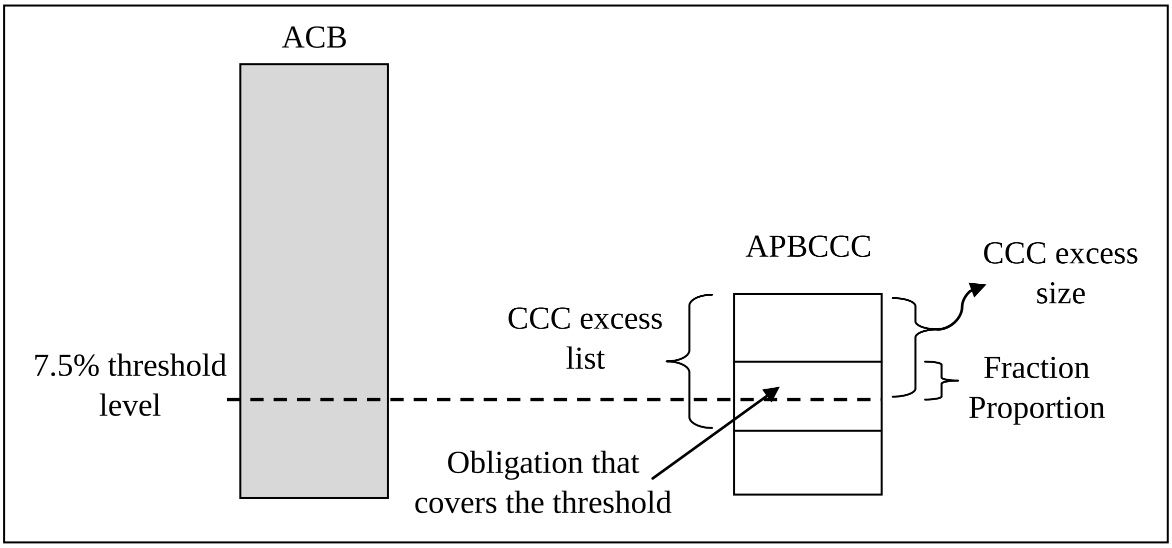

Figure 1. Visual representation of the CCC excess components.

Once this subset is obtained the value of the obligations in the list are multiplied by 1 − *CCCMV* and summed up. For the special obligation that may be covering the threshold, only the fractional proportion is multiplied and considered. The final entire summed value is known as the CCC adjustment value.

For obligations in the category of *isDisc*, the value is summed up multiplied by 1-*PP* and this sum is known as the Discount obligation adjustment.

The Par Coverage value is then the ACB subtracted by the CCC adjustment and Discount obligation adjustment values.

The Weighted Average Spread and Weighted Average Rating Factor are the average Spread, and Rating Factor respectively, weighted by the value of the obligations. Defaulted obligations are not considered when calculating the WAS and WARF.

There also exists an amount of *principal cash* associated with the portfolio for buying obligations. If obligations are sold, then the proceeds are added to the principal cash. It is necessary for there to be enough principal cash if an obligation is to be bought.

2.4 Model Formulation

Indices

i = obligation in list, 1, 2, ..., n.

j = obligation in list, i, i+1, ..., n.

Decision Variables

$$x_i = \begin{cases} 1 & \text{if obligation } i \text{ is in Portfolio} \\ 0 & \text{otherwise} \end{cases}$$ for i = 1, 2, ..., n

$$z_i = \begin{cases} 1 & \text{if obligation } i \text{ is in the CCC excess list, excluding covering obligation} \\ 0 & \text{otherwise} \end{cases}$$
for i = 1, 2, ..., n

$$w_i = \begin{cases} 1 & \text{if obligation } i \text{ is in the CCCexcess list, including covering obligation} \\ 0 & \text{otherwise} \end{cases}$$
for i = 1, 2, ..., n

$xn_i \in \{0, 1, 2, \ldots, M_i\}$, the Number of blocks of obligation i that is in the portfolio
for i = 1, 2, ..., n

$zn_i \in \{0, 1, 2, \ldots, M_i\}$, value of xn_i if obligations i is in z list, 0 otherwise
for i = 1, 2, ..., n

$wn_i \in \{0, 1, 2, \ldots, M_i\}$, value of xn_i if obligations i is in z list, 0 otherwise
for i = 1, 2, ..., n

$$Tsub_i = \begin{cases} \text{FP, if i the threshold obligatio}n \\ \text{0 otherwise} \end{cases}$$
Fractional Proportion of Threshold Obligation Substitute
for i = 1, 2, ..., n

$extra =$ a slack variable preventing negative CCC excess list sizes.

Model

Maximise:

Par Coverage value:

$$
\begin{aligned}
&= \sum_{i=1}^{n} isN_i\, B_i\, xn_i \\
&+ \sum_{i=1}^{n} isD_i\, WV_i\, B_i xn_i \\
&+ \left(ipc\; - \sum_{i=1}^{n} (1 - InPort) Buy_i\, B_i xn_i \right. \\
&+ \left. \sum_{i=1}^{n} (InPort) Sell_i\, B_i (M_i - \boldsymbol{xn}_i) \right) + \sum_{i=1}^{n} isLD_i\, MR_i\, B_i xn_i \\
&+ \sum_{i=1}^{n} isZC_i\, AV_i\, B_i xn_i - \sum_{i=1}^{n} (1 - CCCMV_i)\, B_i zn_i \\
&- \sum_{i=1}^{n} (1 - CCCMV_i) Tsub_i - \sum_{i=1}^{n} isDisc_i\, (1 - PP_i)\, B_i xn_i
\end{aligned}
$$

Subject to:

(1)

$$
\begin{aligned}
&\left(\sum_{i=1}^{n} isCCC_i\, B_i\, xn_i \right) \\
&-0.075 \\
&\times \left(\sum_{i=1}^{n} isN_i\, B_i\, xn_i + \sum_{i=1}^{n} isD_i\, WV_i\, B_i xn_i \right. \\
&+ \left(ipc\; - \sum_{i=1}^{n} (1 - InPort) Buy_i\, B_i xn_i\; + \sum_{i=1}^{n} (InPort) Sell_i\, B_i (M_i - xn_i) \right) \\
&\left. + \sum_{i=1}^{n} isLD_i\, MR_i\, B_i xn_i + \sum_{i=1}^{n} isZC_i\, AV_i\, B_i xn_i \right) \\
&+ extra \geq 0
\end{aligned}
$$

(2) (i)	$xn_i \geq x_i$	for *i* = 1, 2, ... n
(2) (ii)	$xn_i \leq M_i\, x_i$	for *i* = 1, 2, ... n
(3) (i)	$z_i \leq isCCC_i\, x_i$	for *i* = 1, 2, ... n
(3) (ii)	$w_i \leq isCCC_i\, x_i$	for *i* = 1, 2, ... n
(4) (i)	$zn_i \geq z_i$	for *i* = 1, 2, ..., n
(4) (ii)	$zn_i \leq M_i z_i$	for *i* = 1, 2, ..., n

(5) (i) $zn_i \leq xn_i$ for i = 1, 2, ..., n

(5) (ii) $zn_i \geq xn_i - M_i(1 - z_i)$ for i = 1, 2, ..., n

(6) (i) $wn_i \geq w_i$ for i = 1, 2, ..., n

(6) (ii) $wn_i \leq M_i w_i$ for i = 1, 2, ..., n

(7) (i) $wn_i \leq xn_i$ for i = 1, 2, ... n

(7) (ii) $wn_i \geq xn_i - M(1 - w_i)$ for i = 1, 2, ..., n

(8) (i)

$$\sum_{i=1}^{n} B_i\ zn_i \leq \left(\sum_{i=1}^{n} isCCC_i\ B_i\ xn_i\right) - 0.075 \times \left(\sum_{i=1}^{n} isN_i\ B_i\ xn_i + \sum_{i=1}^{n} isD_i\ WV_i\ B_i xn_i + \left(ipc - \sum_{i=1}^{n}(1 - InPort)Buy_i\ B_i xn_i + \sum_{i=1}^{n}(InPort)Sell_i\ B_i(M_i - xn_i)\right) + \sum_{i=1}^{n} isLD_i\ MR_i\ B_i xn_i + \sum_{i=1}^{n} isZC_i\ AV_i\ B_i xn_i\right) + extra$$

(8) (ii)

$$\sum_{i=1}^{n} B_i\ wn_i \geq \left(\sum_{i=1}^{n} isCCC_i\ B_i\ xn_i\right) - 0.075 \times \left(\sum_{i=1}^{n} isN_i\ B_i\ xn_i + \sum_{i=1}^{n} isD_i\ WV_i\ B_i xn_i + \left(ipc - \sum_{i=1}^{n}(1 - InPort)Buy_i\ B_i xn_i + \sum_{i=1}^{n}(InPort)Sell_i\ B_i(M_i - xn_i)\right) + \sum_{i=1}^{n} isLD_i\ MR_i\ B_i xn_i + \sum_{i=1}^{n} isZC_i\ AV_i\ B_i xn_i\right) + extra$$

(9) (i)

$$\sum_{i=1}^{n} z_i \leq \sum_{i=1}^{n} w_i$$

(9) (ii)

$$\left(\sum_{i=1}^{n} z_i\right) + 1 \geq \sum_{i=1}^{n} w_i$$

(10) (i) $z_j \leq 1 - (x_i - z_i)$

$i = 1, 2, ..., n, \quad j = i, i+1, ..., n$

for *i* and *j* that are *isCCC*

(10) (ii) $w_j \leq 1 - (x_i - w_i)$

$i = 1, 2, ..., n, \quad j = i, i+1, ..., n$

for *i* and *j* that are *isCCC*

(11)

$$Tsub_i \geq \left(\sum_{i=1}^{n} isCCC_i\, B_i\, xn_i\right) - 0.075 \times \left(\sum_{i=1}^{n} isN_i\, B_i\, xn_i + \sum_{i=1}^{n} isD_i\, WV_i\, B_i xn_i + \left(ipc - \sum_{i=1}^{n}(1 - InPort)Buy_i\, B_i xn_i + \sum_{i=1}^{n}(InPort)Sell_i\, B_i(M_i - xn_i)\right) + \sum_{i=1}^{n} isLD_i\, MR_i\, B_i xn_i + \sum_{i=1}^{n} isZC_i\, AV_i\, B_i xn_i\right) + extra - \left(\sum_{i=1}^{n} B_i\, zn_i\right) - B_i M_i \times (1 - (w_i - z_i))$$

for $i = 1, 2, ..., n$

(12)

$$\sum_{i=1}^{n}(1 - isD_i)\ spread_i B_i xn_i \geq minimum\ WAS \times \sum_{i=1}^{n}(1 - isD_i)\ B_i xn_i$$

(13)

$$\sum_{i=1}^{n}(1 - isD_i)\ RF_i B_i xn_i \leq maximum\ WARF \times \sum_{i=1}^{n}(1 - isD_i)\ B_i xn_i$$

(14)

$$ipc - \sum_{i=1}^{n} (1 - inPort_i)\, Buy_i\, B_i xn_i + \sum_{i=1}^{n} inPort_i Sell_i\, B_i(M_i - xn_i) \geq minimum\ cash$$

3 Results

The formulation was implemented in the Python programming language, through the PuLP optimisation interface. Various solvers were tested, however the following results were obtained from the CBC solver.

Considering the inverse correlation between the Par Coverage and the WAS value, and the fact that it would also be beneficial to consider portfolios with high WAS values, an efficient frontier could be created by maximising the Par Coverage value multiple times, incrementing the *minimum WAS* parameter each time (see Figure 2).

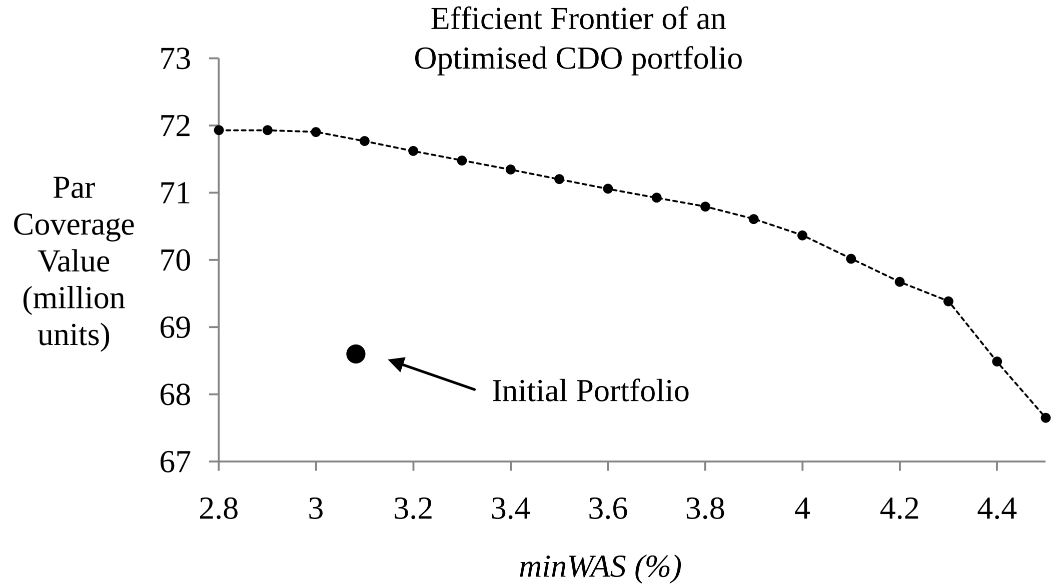

Figure 2. Results from a small test CDO portfolio constructed out of real data provided by M&G Investments. Values for real investment CDO portfolios were omitted as the data is confidential, however were tested and show similar results.

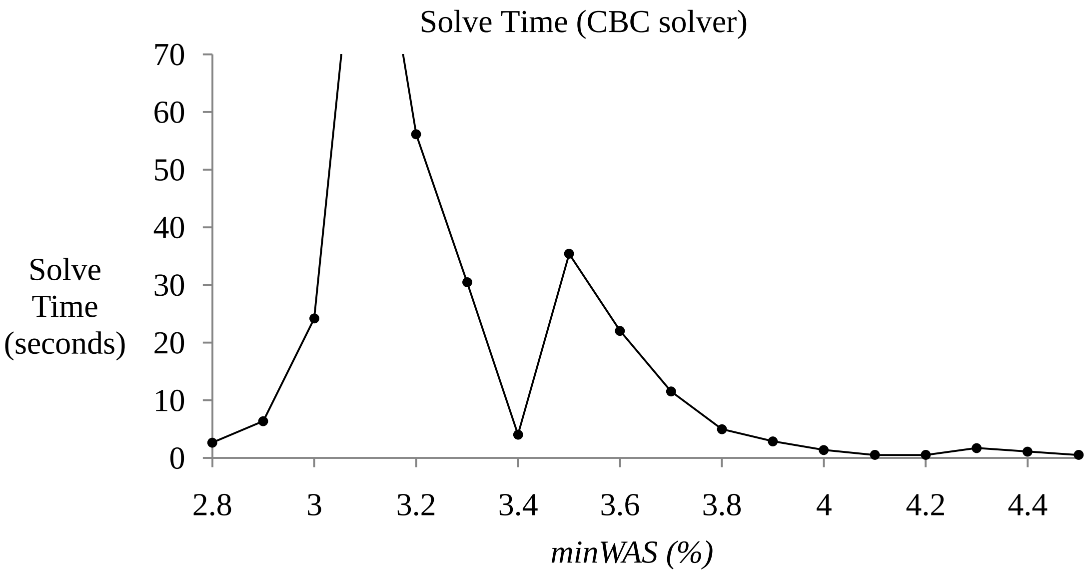

Figure 3. Solve Time resulted from obtaining the results in Figure 2.

The Solve speed was found to be highly variable and was a major issue with certain portfolios taking a considerable amount of time to solve, or unsolvable due to exceeding computer memory. This can be seen in for the small test portfolio with a value of around 69 million (currency) units solved in Figure 3, near a *minimum WAS* of 3.1. A typical CDO portfolios are much larger with values ranging in the hundreds of millions of units

The variability in solve time was found to be due to the consideration of CCC obligations. For low *minimum WAS* (i.e., in Figure 3, near 3% and below), the CCC obligations are removed in order to minimise the CCC excess list to 0. For high *minimum WAS* (i.e., near 3.6% and higher), CCC obligations were mostly included in the portfolio, also resulting in fast solve speeds. In between however, the solver must make a lot of considerations between all the CCC obligations against the rest of the portfolio, and this resulted in a very difficult problem to solve.

4 Conclusion

The combined calculations of the Par Coverage value, WAS value, and WARF value were successfully converted into a mixed integer linear programme(MILP) that could be solved in a MILP solver. From the results of the numerous test inputs, general observations could be concluded; with the Par Coverage value as the objective function, there was a tendency for the MILP to attempt to minimise excess CCC category obligations.

Due to the inverse relationship of the value of obligations and the spread of obligations, an efficient frontier could be obtained in the results. For all test problems that were solved, the MILP was able to identify an efficient frontier of portfolios with higher Par Coverage and WAS up to this frontier.

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Note: Internal documents provided by M&G Investments were consulted and are confidential in nature and thus are not referenced.